Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 346 (2008) 963–968
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Partial Differential Equations

Periodic unfolding and nonhomogeneous Neumann problems in domains with small holes

Amar Ould Hammouda [a,b]

[a] *Laboratoire Jacques-Louis Lions-CNRS, boîte courrier 187, Université Pierre-et-Marie-Curie, 4 place Jussieu, 75005 Paris, France*
[b] *Departement of Mathematics, ENS, P.O. Box 92, 16050 Kouba, Algiers*

Received 9 May 2008; accepted 16 June 2008

Available online 13 August 2008

Presented by Philippe G. Ciarlet

Abstract

We consider elliptic problems in periodically perforated domains in $\mathbb{R}^N$, with nonhomogeneous Neumann conditions on the boundary of the holes. We are interested in the asymptotic behavior of the solutions as the period ε goes to zero. In a first case all the holes are "small", i.e., are of size $r(\varepsilon)$ with $r(\varepsilon)/\varepsilon \to 0$. In the second case, there are again small holes but also holes of size ε. We use the periodic unfolding method introduced in Cioranescu et al. (2002), which allows us to study second order operators with highly oscillating coefficients and so, to generalize here the results of Conca and Donato (1988). In both cases, if $r(\varepsilon) = \exp(N/N-1)$, an additional term appears in the right-hand side of the limit equation. ***To cite this article: A. Ould Hammouda, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Éclatement périodique et problèmes de Neumann non homogènes dans des domaines à petits trous. L'objet de cette Note est l'homogénéisation d'une classe de problèmes élliptiques dans des domaines de $\mathbb{R}^N$, périodiquement perforés par des petits trous, avec des conditions de Neumann non homogènes sur le bord des trous. Dans un premier temps, les trous de taille $r(\varepsilon)$ avec $r(\varepsilon)/\varepsilon \to 0$ et dans un second, on a des trous de taille $r(\varepsilon)$ mais aussi des trous de taille ε. Le premier cas, pour le Laplacien, a été étudié dans Conca et Donato (1988). Pour étudier le comportement asymptotique des solutions lorsque $\varepsilon \to 0$, on utilise ici la méthode de l'éclatement périodique introduite par Cioranescu et al. (2002), ce qui permet de considérer des opérateurs de second ordre à coefficients oscillants. Dans les deux situations, pour $r(\varepsilon) = \exp(N/N-1)$, on a un terme supplémentaire qui apparait dans le second membre de l'équation limite. ***Pour citer cet article : A. Ould Hammouda, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Nous adaptons ici, la méthode de l'éclatement périodique introduite dans [2], à une classe de problèmes élliptiques posés sur des domaines de $\mathbb{R}^N$, périodiquement perforés par des petits trous, avec des conditions de Neumann non

E-mail address: amar.ouldhamouda@ens-kouba.dz.

1631-073X/$ – see front matter © 2008 Published by Elsevier Masson SAS on behalf of Académie des sciences.
doi:10.1016/j.crma.2008.07.001

homogènes sur le bord des trous. La méthode (voir [2]), utilise un opérateur d'éclatement périodique et une décomposition macro-micro des fonctions, séparant les échelles.

Soit Ω un domaine borné de $\mathbb{R}^N$, $N \geqslant 3$ tel que $|\partial\Omega| = 0$ et $Y = [0,1[^N$ la cellule de référence. Pour définir le domaine perforé $\Omega_{\varepsilon,\delta}$, de frontiére $\partial\Omega_{\varepsilon,\delta}$ lipschitzienne, on introduit les ensembles suivants : B et T sont deux compacts de Y tel que $\Theta_\varepsilon = \{k = (k_1, k_2, \dots, k_n)$: $(\varepsilon Y + \varepsilon k) \cap \Omega \neq \phi\}$; $Y_\varepsilon = \bigcup_{k\in\Theta_\varepsilon}\{\varepsilon(Y+k)\}$; $B_{\varepsilon\delta} = \bigcup_{k\in\Theta_\varepsilon}\{\varepsilon\delta B + \varepsilon k\}$; et $\Omega_{\varepsilon\delta} = \Omega \setminus B_{\varepsilon\delta}$. La principale caracteristique de $\Omega_{\varepsilon\delta}$ est que la dimension des trous n'est pas nécessairement proportionnelle à celle de la cellule εY. Enfin, soit $Y_\delta = Y \setminus \delta\bar{B}$; de plus, on pose $\theta = \frac{|Y\setminus T|}{|Y|}$.

1. Perforated domains

The periodic unfolding method was introduced in [2] by Cioranescu, Damlamian and Griso for the study of periodic homogenization in the case of fixed domains. It is based on two ingredients: the unfolding operator and a macro-micro decomposition of functions, allowing to separate the macroscopic and microscopic scales. This method, being a fixed-domains one, no extension operator is needed and avoids any construction of special test functions. Consequently, we can consider a larger class of geometrical situations than in [1,3,5]. We use this method here in order to treat elliptic problems in domains with small holes and nonhomogeneous Neumann conditions on the boundary of the holes.

In the sequel, ε and δ are two small parameters going to zero. We start by defining two perforated domains, $\Omega_{\varepsilon\delta}$ and $\Omega^\star_{\varepsilon\delta}$. To do so, let Ω be a bounded domain in $\mathbb{R}^N$, $(N \geqslant 3)$ such that $|\partial\Omega| = 0$ and let $Y = [0,1[^N$ be the reference (or periodicity) cell. We introduce the following notation:

$$\hat{\Omega}_\varepsilon = \text{interior}\Big\{\bigcup_{\xi\in\Xi_\varepsilon} \varepsilon(\xi + \bar{Y})\Big\}, \quad \text{where } \Xi_\varepsilon = \big\{\xi \in \mathbb{Z}^N,\ \varepsilon(\xi + Y) \subset \Omega\big\}, \tag{1}$$

and set $\Lambda_\varepsilon = \Omega \setminus \hat{\Omega}_\varepsilon$. The set $\hat{\Omega}_\varepsilon$ is the largest finite union of εY cells contained in Ω, and Λ_ε is the subset of Ω containing the parts from εY cells intersecting $\partial\Omega$.

Case 1. Let B be an open set, such that $B \Subset Y$, this is the hole in Y. Denote $Y_\delta = Y \setminus \delta\bar{B}$ supposed to be connected. Set

$$B_{\varepsilon\delta} = \bigcup_{\xi\in\mathbb{Z}^n} \varepsilon(\xi + \delta B), \qquad \Omega_{\varepsilon\delta} = \left\{x \in \Omega \;\middle|\; \left\{\frac{x}{\varepsilon}\right\} \in Y_\delta\right\}, \tag{2}$$

where $B_{\varepsilon\delta}$ is the set of ε-periodic holes of size $\varepsilon\delta$ in $\mathbb{R}^N$, and $\Omega_{\varepsilon\delta} = (\mathbb{R}^N \setminus B_{\varepsilon\delta}) \cap \Omega$ is the perforated domain, the holes being of size $\varepsilon\delta$. We denote by $B^{\text{int}}_{\varepsilon\delta}$ the set of holes in Ω that do not meet the boundary $\partial\Omega$. In the sequel, n^B_ε denotes the unit outward normal vector to $B_{\varepsilon\delta}$. By construction (see (2)), n_ε is actually equal to n^B, the unit outward normal vector to B.

Case 2. Let T be another open set, $T \Subset Y$ and such that $B \cap T = \emptyset$. The part corresponding to the material in the cell Y is now $Y^\star_\delta = Y \setminus (\bar{T} \cup \overline{\delta B})$; it is assumed to be connected and set

$$T_\varepsilon = \bigcup_{\xi\in\mathbb{Z}^n} \varepsilon(\xi + T). \tag{3}$$

The perforated domain $\Omega^\star_{\varepsilon\delta}$ with ε-periodic perforations of size $\varepsilon\delta$ and ε-periodic perforations of size ε in the same time, is obtained by removing from Ω the set of holes $B_{\varepsilon\delta}$ and T_ε,

$$\Omega^\star_{\varepsilon\delta} = \Omega \setminus (B_{\varepsilon\delta} \cup T_\varepsilon) = \left\{x \in \Omega \;\middle|\; \left\{\frac{x}{\varepsilon}\right\} \in Y^\star_\delta\right\}. \tag{4}$$

As above, n^T_ε denotes the unit outward normal vector to $B_{\varepsilon\delta}$. By construction (see (2)), n_ε is actually equal to n, the unit outward normal vector to B.

We will also use in the sequel the notation:

$$\begin{cases} Y^* = Y \setminus \bar{T}, \quad \theta = \dfrac{|Y^\star|}{|Y|}, \\ \hat{\Omega}^*_\varepsilon = \hat{\Omega}_\varepsilon \setminus \bar{T}_\varepsilon = \left\{x \in \Omega \;\middle|\; \left\{\dfrac{x}{\varepsilon}\right\} \in Y^*\right\}, \quad \Lambda^*_\varepsilon = \Omega^*_\varepsilon \setminus \hat{\Omega}^*_\varepsilon, \end{cases} \tag{5}$$

where $\hat{\Omega}_\varepsilon^*$ is a perforated domain with ε-periodic perforations of size ε. As in Case 1, $T_\varepsilon^{\mathrm{int}}$ denotes the set of holes in T_ε, that do not meet the boundary $\partial\Omega$.

2. Unfolding operators in perforated domains and boundary unfolding operators

Following [2], $[z]$ denotes the unique integer combination $\sum_{j=1}^N n_j\ell_j$ such that $z-[z]_Y$ belongs to Y and set $\{z\}_Y = z-[z]_Y$. For $x\in\mathbb{R}^N$, there exists a unique element $[\frac{x}{\varepsilon}]_Y$ such that $x-\varepsilon[\frac{x}{\varepsilon}]_Y=\varepsilon\{\frac{x}{\varepsilon}\}_Y$, where $\{\frac{x}{\varepsilon}\}_Y\in Y$. For domains without holes, the definition of the *periodic unfolding operator* introduced in [2] is the following:

$$\mathcal{T}_\varepsilon(\phi)(x,y)=\begin{cases}\phi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y+\varepsilon y\right) & \text{a.e. for } (x,y)\in\hat{\Omega}_\varepsilon\times Y,\\ 0 & \text{a.e. for } (x,y)\in\Lambda_\varepsilon\times Y,\end{cases}\tag{6}$$

for any ϕ Lebesgue-measurable on Ω. This operator acts from $L^p(\Omega)$ to $L^p(\Omega\times Y)$.

Let us recall the main properties of $\mathcal{T}_\varepsilon$ from [2] (for proofs, we refer the reader to [2] and [6]):

Proposition 2.1. *If $\{w_\varepsilon\}$ is a sequence in $L^1(\Omega)$ satisfying $\int_{\Lambda_\varepsilon}|w_\varepsilon|\,\mathrm{d}x\to 0$. Then*

$$\int_\Omega w_\varepsilon\,\mathrm{d}x\overset{\mathcal{T}_\varepsilon}{\simeq}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(w_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\quad\textit{i.e.,}\quad\int_\Omega w_\varepsilon\,\mathrm{d}x-\int_{\Omega\times Y}\mathcal{T}_\varepsilon(w_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\to 0.$$

Proposition 2.2. *Let $w_\varepsilon\rightharpoonup w$ weakly in $H^1(\Omega)$. Then, up to a subsequence, there exists $\hat{w}\in L^2(\Omega;H^1_{\mathrm{per}}(Y))$ such that*

$$\mathcal{T}_\varepsilon(\nabla w_\varepsilon)\rightharpoonup\nabla_x w+\nabla_y\hat{w}\quad\textit{weakly in } L^2(\Omega\times Y).$$

For domains with holes we have a definition similar to (6), see for more details [4]. For ϕ Lebesgue-measurable on $\hat{\Omega}_\varepsilon$, the *periodic unfolding operator* $\mathcal{T}_\varepsilon^*$ is defined by

$$\mathcal{T}_\varepsilon^*(\phi)(x,y)=\begin{cases}\phi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y+\varepsilon y\right) & \text{a.e. for } (x,y)\in\hat{\Omega}_\varepsilon\times Y^*,\\ 0 & \text{a.e. for } (x,y)\in\Lambda_\varepsilon\times Y^*.\end{cases}\tag{7}$$

Let us observe that $\mathcal{T}_\varepsilon^*(\phi)=\mathcal{T}_\varepsilon(\tilde{\phi})|_{\Omega\times Y^*}$, so that the operator $\mathcal{T}_\varepsilon^*$ has almost the same properties as the operator $\mathcal{T}_\varepsilon$. In particular, one has the following results, corresponding respectively, to Propositions 2.1 and 2.2:

Proposition 2.3. *If $\{w_\varepsilon\}$ is a sequence in $L^1(\Omega_\varepsilon^*)$ satisfying $\int_{\Lambda_\varepsilon^*}|w_\varepsilon|\,\mathrm{d}x\to 0$, then*

$$\int_{\Omega_\varepsilon^*} w_\varepsilon\,\mathrm{d}x\overset{\mathcal{T}_\varepsilon^*}{\simeq}\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon^*(w_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\quad\textit{i.e.,}\quad\int_{\Omega_\varepsilon^*} w_\varepsilon\,\mathrm{d}x-\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon^*(w_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\to 0.$$

Proposition 2.4. *Let w_ε belong to $H^1(\Omega_\varepsilon^*)$ and satisfying $\|w_\varepsilon\|_{H^1(\Omega_\varepsilon^*)}\leqslant C$. Then, up to a subsequence, there exist w in $H^1(\Omega)$ and $\hat{w}$ in $L^2(\Omega;H^1_{\mathrm{per}}(Y^*))$, such that*

$$\widetilde{w_\varepsilon}\rightharpoonup\theta w\quad\textit{weakly in } L^2(\Omega),$$
$$\mathcal{T}_\varepsilon^*(w_\varepsilon)\rightharpoonup w\quad\textit{weakly in } L^2\big(\Omega;H^1(Y^*)\big),$$
$$\mathcal{T}_\varepsilon^*(\nabla w_\varepsilon)\rightharpoonup\nabla w+\nabla_y\hat{w}\quad\textit{weakly in } L^2(\Omega\times Y^*).$$

Now, we recall the definition of the operator $\mathcal{T}_{\varepsilon\delta}^b$, a linear unfolding operator on the boundary of the holes $B_{\varepsilon\delta}$, specific for small holes.

Definition 2.5. Let $\phi \in L^p(\partial B_{\varepsilon\delta})$, with $p \in [1,+\infty[$. The boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta}$ is defined by:

$$\mathcal{T}^b_{\varepsilon\delta}(\phi)(x,z) = \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon\delta z\right) \quad \text{a.e. for } x \in \mathbb{R}^n,\ z \in \partial B. \tag{8}$$

It is easily seen that for every ϕ in $L^1(\partial B_{\varepsilon\delta})$,

$$\int_{\partial B_{\varepsilon\delta}} \phi(x)\,\mathrm{d}\sigma(x) = \frac{\delta^{N-1}}{\varepsilon} \int_{\mathbb{R}^N\times\partial B} \mathcal{T}^b_{\varepsilon\delta}(\phi)(x,z)\,\mathrm{d}x\,\mathrm{d}\sigma(z).$$

For g in $L^2(\partial B)$, denote by $\mathcal{M}_{\partial B}(g)$ its mean value on ∂B, $\mathcal{M}_{\partial B}(g) = \frac{1}{|\partial B|}\int_{\partial B} g\,\mathrm{d}\sigma$.

Proposition 2.6. *Let $g \in L^2(\partial B)$ and set*:

$$g_{\varepsilon\delta}(x) = g\left(\frac{1}{\delta}\left(\frac{x}{\varepsilon}\right)\right) \quad \textit{for all } x \in \partial B_{\varepsilon\delta}. \tag{9}$$

The following estimate holds for every $\phi \in H^1(\Omega)$:

$$\left|\int_{\partial B_{\varepsilon\delta}} g_{\varepsilon\delta}(x)\phi\,\mathrm{d}x\right| \leqslant C\frac{\delta^{n-1}}{\varepsilon}\left(\left|\mathcal{M}_{\partial B}(g)\right| + \varepsilon\delta\right)\|\nabla\phi\|_{(L^2(\Omega))^N}. \tag{10}$$

Moreover, as $\varepsilon \to 0$ one has the convergences:

1. *If $\mathcal{M}_{\partial B}(g) \neq 0$, then $\frac{\varepsilon}{\delta^{N-1}}\int_{\partial B_{\varepsilon\delta}} g_{\varepsilon\delta}(x)\phi(x)\,\mathrm{d}\sigma(x) \to |\partial B|\mathcal{M}_{\partial B}(g)\int_\Omega \phi(x)\,\mathrm{d}x$;*
2. *If $\mathcal{M}_{\partial B}(g) = 0$, then $\int_{\partial B_{\varepsilon\delta}} g_{\varepsilon\delta}(x)\phi(x)\,\mathrm{d}\sigma(x) \to 0$.*

Idea of the proof. For holes of size of order of ε (i.e., with $\delta = 1$), a boundary operator denoted $\mathcal{T}^b_\varepsilon$, was introduced for the first time in [3], its definition is (8) with $\delta = 1$. Most of the properties of $\mathcal{T}^b_{\varepsilon\delta}$ are almost transcriptions of the corresponding ones of $\mathcal{T}^b_\varepsilon$ and are obtained by a rescaling in δ (for details, see [7]).

3. The main homogenization results

Let $A^\varepsilon(x) = (a^\varepsilon_{ij}(x))_{1\leqslant i,j\leqslant N}$ be a measurable matrix, bounded in $L^\infty(\Omega)$ and satisfying:

$$\alpha|\xi|^2 \leqslant A^\varepsilon(x)\xi\xi \leqslant \beta|\xi|^2 \quad \text{a.e. } x \in \Omega,\ \text{with } \alpha > 0,\ \beta > 0. \tag{11}$$

Let us assume that there exists a constant k satisfying

$$k = \lim_{\varepsilon\to 0}\frac{\delta^{N-1}}{\varepsilon}, \quad \text{with } 0 \leqslant k < \infty. \tag{12}$$

Problem 1. With the geometry and notation described in Case 1 from Section 2, consider the following problem:

$$\begin{cases} -\mathrm{div}(A^\varepsilon\nabla u_{\varepsilon\delta}) = f & \text{in } \Omega_{\varepsilon\delta},\\ A^\varepsilon\nabla u_{\varepsilon\delta}\cdot n^B_\varepsilon = g_{\varepsilon\delta} & \text{on } \partial B^{\mathrm{int}}_{\varepsilon\delta},\\ u_{\varepsilon\delta} = 0 & \text{on } \partial\Omega_{\varepsilon\delta}\setminus\partial B^{\mathrm{int}}_{\varepsilon\delta}, \end{cases} \tag{13}$$

where $f \in L^2(\Omega)$ and $g_{\varepsilon\delta}$ is defined by (9) with g in $L^2(\partial B)$.

Problem 2. With the geometry described in Case 2 from Section 2, consider the problem:

$$\begin{cases} -\mathrm{div}(A^\varepsilon\nabla u^*_{\varepsilon\delta}) = f & \text{in } \Omega^*_{\varepsilon\delta},\\ A^\varepsilon\nabla u^*_{\varepsilon\delta}\cdot n^T_\varepsilon = h_\varepsilon & \text{on } \partial T^{\mathrm{int}}_\varepsilon,\\ A^\varepsilon\nabla u^*_{\varepsilon\delta}\cdot n^B_\varepsilon = g_{\varepsilon\delta} & \text{on } \partial B^{\mathrm{int}}_{\varepsilon\delta},\\ u^*_{\varepsilon\delta} = 0 & \text{on } \partial\Omega^*_{\varepsilon\delta}\setminus(\partial B^{\mathrm{int}}_{\varepsilon\delta}\cup\partial T^{\mathrm{int}}_\varepsilon), \end{cases} \tag{14}$$

where $h^\varepsilon(x) = h(\frac{x}{\varepsilon})$ with h in $L^2(\partial T)$.

Theorem 3.1. (*Problem 1.*) *Suppose that* (11) *and* (12) *are satisfied. Let us assume that*

$$\mathcal{T}_\varepsilon(A^\varepsilon)(x,y) \to A(x,y) \quad \text{a.e. in } \Omega \times Y. \tag{15}$$

Let $u_{\varepsilon\delta}$ be the solution of (13). *There exist u_0 in $H_0^1(\Omega)$ and $\hat{u}$ in $L^2(\Omega; H^1_{\mathrm{per}}(Y))$ with*

$$\begin{aligned} &\tilde{u}_{\varepsilon\delta} \rightharpoonup u_0 \quad \text{weakly in } L^2(\Omega), \\ &\mathcal{T}_\varepsilon(u_{\varepsilon\delta}) \rightharpoonup u_0 \quad \text{weakly in } L^2\big(\Omega; H^1_{\mathrm{loc}}(Y)\big), \end{aligned} \tag{16}$$

and such that, for all Ψ in $H_0^1(\Omega)$ and for all Φ in $L^2(\Omega; H^1_{\mathrm{per}}(Y))$, one has:

$$\int_{\Omega\times Y} A(\nabla_x u_0 + \nabla_y \hat{u})(\nabla\psi + \nabla_y \Phi)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f\psi\,\mathrm{d}x + k|\partial B|\mathcal{M}_{\partial B}(g)\int_\Omega \Psi\,\mathrm{d}x. \tag{17}$$

Theorem 3.2. *(Problem 2.) Under the same assumptions as in Theorem* 3.1*, let $u^*_{\varepsilon\delta}$ be the solution of* (14).

(i) *Let us assume that $\mathcal{M}_{\partial T}(h) \neq 0$. Then, there exist $u_0^* \in H_0^1(\Omega)$, $\hat{u}^* \in L^2(\Omega; H^1_{\mathrm{per}}(Y))$ with*

$$\begin{aligned} &\varepsilon\tilde{u}_{\varepsilon\delta} \rightharpoonup \theta u_0 \quad \text{weakly in } L^2(\Omega), \\ &\mathcal{T}^*_\varepsilon(\varepsilon u_{\varepsilon\delta}) \rightharpoonup u_0 \quad \text{weakly in } L^2\big(\Omega; H^1_{\mathrm{loc}}(Y^\star)\big), \end{aligned}$$

and satisfying, for all $\Psi \in H_0^1(\Omega)$ and for all Φ in $L^2(\Omega; H^1_{\mathrm{per}}(Y^\star))$,

$$\int_{\Omega\times Y^\star} A(\nabla_x u_0^* + \nabla_y \hat{u}_0^*)(\nabla\Psi + \nabla_y \Phi)\,\mathrm{d}x\,\mathrm{d}y = \theta\int_\Omega f\Psi\,\mathrm{d}x + \theta|\partial T|\mathcal{M}_{\partial T}(h)\int_\Omega \Psi(x)\,\mathrm{d}x.$$

(ii) *Suppose that $\mathcal{M}_{\partial T}(h) = 0$. Then, there exist $u^* \in H_0^1(\Omega)$, $\hat{u}^* \in L^2(\Omega; H^1_{\mathrm{per}}(Y))$ with*

$$\begin{aligned} &\tilde{u}_{\varepsilon\delta} \rightharpoonup \theta u_0 \quad \text{weakly in } L^2(\Omega), \\ &\mathcal{T}^*_\varepsilon(u_{\varepsilon\delta}) \rightharpoonup u_0 \quad \text{weakly in } L^2\big(\Omega; H^1_{\mathrm{loc}}(Y^\star)\big), \end{aligned}$$

and satisfying, for all $\Psi \in H_0^1(\Omega)$ and for all Φ in $L^2(\Omega; H^1_{\mathrm{per}}(Y^\star))$,

$$\int_{\Omega\times Y^\star} A(\nabla_x u^* + \nabla_y \hat{u}^*)(\nabla\Psi + \nabla_y \Phi)\,\mathrm{d}x\,\mathrm{d}y = \theta\int_\Omega f\Psi\,\mathrm{d}x + k\theta|\partial B|\mathcal{M}_{\partial B}(g)\int_\Omega \Psi\,\mathrm{d}x.$$

Sketch of the proof of Theorem 3.1. Define the following functional space:

$$V_0^{\varepsilon\delta} = \big\{v \in H^1(\Omega_{\varepsilon\delta}) \mid v = 0 \text{ on } \partial\Omega \cap \partial\Omega_{\varepsilon\delta}\big\},$$

which a Hilbert space for the norm of the gradient. The variational formulation of Problem 1 is: find $u_{\varepsilon\delta}$ in $V_0^{\varepsilon\delta}$ satisfying:

$$\int_{\Omega_{\varepsilon\delta}} A^\varepsilon \nabla u_{\varepsilon\delta}\nabla\phi\,\mathrm{d}x = \int_{\Omega_{\varepsilon\delta}} f\phi\,\mathrm{d}x + \int_{\partial B_{\varepsilon\delta}} g_{\varepsilon\delta}\,\phi\,\mathrm{d}s, \quad \forall\phi \in V_{\varepsilon\delta}. \tag{18}$$

Then, due to properties (11) of the operator A^ε, by Lax–Milgram theorem, there exists $u_{\varepsilon\delta}$ in $V_{\varepsilon\delta}$, unique solution of Problem 1. By taking $u_{\varepsilon\delta}$ as a test function in (18), thanks to Proposition 2.6, we get immediately the estimate:

$$\|u_{\varepsilon\delta}\|_{V_0^{\varepsilon\delta}} \leqslant C,$$

uniformly with respect to ε and δ. Then convergences (16) follow from Proposition 2.2 which also gives the existence of $\hat{u} \in L^2(\Omega; H^1_{\mathrm{per}}(Y))$ such that (up to a subsequence),

$$\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}) \rightharpoonup \nabla_x u_0 + \nabla_y \hat{u} \quad \text{weakly in } L^2\big(\Omega; L^2_{\mathrm{loc}}(Y)\big). \tag{19}$$

Let now ϕ in $\mathcal{D}(\Omega)$. For ε and δ small enough, its restriction to $\Omega_{\varepsilon\delta}$ is in $V_0^{\varepsilon\delta}$ and so, it can be taken as test function in (18). Unfolding the left-hand side term in (18) with $\mathcal{T}_\varepsilon$, one gets:

$$\int_{\Omega\times Y_\delta} \mathcal{T}_\varepsilon(A^\varepsilon)(x,y)\,\mathcal{T}_\varepsilon(\nabla_x u_{\varepsilon,\delta})(x,y)\nabla\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$\overset{\mathcal{T}_\varepsilon}{\simeq} \int_{\Omega_{\varepsilon\delta}} f\phi\,\mathrm{d}x + \frac{\delta^{N-1}}{\varepsilon}\int_{\mathbb{R}^N\times\partial B} g(z)\mathcal{T}^b_{\varepsilon\delta}(\phi)(x,z)\,\mathrm{d}x\,\mathrm{d}\sigma(z). \tag{20}$$

Using convergences (15) and (19) as well as Proposition 2.6, one immediately gets:

$$\int_{\Omega\times Y} A(\nabla_x u_0+\nabla_y\hat{u})\nabla\phi\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f\psi\,\mathrm{d}x + k\int_{\partial B} g(z)\,\mathrm{d}\sigma_z\int_\Omega \phi\,\mathrm{d}x, \tag{21}$$

which, by density, holds for any ϕ in $H_0^1(\Omega)$. The next step is to take as test function in (18), $w(\cdot)=\varepsilon\psi(\cdot)\phi(\frac{\cdot}{\varepsilon})$, with $\psi\in\mathcal{D}(\Omega)$, $\phi\in H^1_{\mathrm{per}}(Y)$. Unfolding again with $\mathcal{T}_\varepsilon$ and passing to the limit, yields:

$$\int_{\Omega\times Y} A(\nabla_x u_0+\nabla_y\hat{u})\psi\nabla_y\phi\,\mathrm{d}x\,\mathrm{d}y=0,$$

which together with (21) gives (17).

Sketch of the proof of Theorem 3.2. The proof in this case follows the along the lines the former one. The major difference is that now, one has to unfold with $\mathcal{T}_\varepsilon^*$ and make use of Propositions 2.3 and 2.4 to get the result.

Remark 3.3. Suppose that A^ε is defined by $A^\varepsilon(\cdot)=A(\cdot/\varepsilon)$, $A(y)=(a_{ij}(y))_{1\leqslant i,j\leqslant N}$ with a_{ij} Y-periodic and A satisfying a.e. on Y an inequality of type (11). Then the unfolding homogenized limit problems from above theorems, can be easily formulated in the standard strong formulation. For example, the limit problem in Theorem 3.1 rewrites in the form:

$$\begin{cases} -\mathrm{div}(\mathcal{A}^{\mathrm{hom}}\nabla u_0)=f+k|\partial B|\mathcal{M}_{\partial B}(g) & \text{in } \Omega,\\ u_0=0 \quad \text{in } \partial\Omega, \end{cases}$$

where $\mathcal{A}^{\mathrm{hom}}$ is the classical homogenized operator for fixed domains:

$$(\mathcal{A}^{\mathrm{hom}})_{ij}=\int_Y\left(a_{ij}(y)-\sum_{k=1}^N a_{ik}(y)\frac{\partial\hat{\chi}_j}{\partial y_k}(y)\right)\mathrm{d}y,$$

with the correctors $\hat{\chi}_j$ $(j=1,\dots,N)$ defined, for all $\phi\in H^1_{\mathrm{per}}(Y)$, by the cell problems,

$$\hat{\chi}_j \quad Y\text{-periodic},\quad \mathcal{M}_Y(\hat{\chi}_j)=0,\quad \int_Y A(y)\nabla(\hat{\chi}^j-y_j)\nabla\phi(y)\,\mathrm{d}y=0.$$

References

[1] A. Bensoussan, J.-L. Lions, G. Papanicolaou, Asymptotic Analysis for Periodic Structures, North-Holland, Amsterdam, 1978.
[2] D. Cioranescu, A. Damlamian, G. Griso, Periodic unfolding and homogenization, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 99–104.
[3] C. Conca, P. Donato, Non-homogeneous Neumann problems in domains with small holes, RAIRO Modél. Math. Anal. Numér. 4 (22) (1988) 561–608.
[4] D. Cioranescu, P. Donato, R. Zaki, The periodic unfolding method in perforated domains, Portugaliae Mathematica 63 (4) (2006) 467–496.
[5] D. Cioranescu, F. Murat, Un terme étrange venu d'ailleurs, in: H. Brezis, J.L. Lions (Eds.), Nonlinear Partial Differential Equations and their Applications, College de France Seminar, II & III, in: Research Notes in Math., vols. 60–70, Pitman, Boston, 1982, pp. 98–138, 154–178.
[6] A. Damlamian, An elementary introduction to periodic unfolding, in: A. Damlamian, D. Lukkassen, A. Meidell, A. Piatnitski (Eds.), Proc. of the Narvik Conference 2004, in: Gakuto Int. Series, Math. Sci. Appl., vol. 24, Gakkokotosho, 2006, pp. 119–136.
[7] A. Ould Hammouda, Homogenization of a class of Neumann problems in perforated domains, in press.