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Probability Theory

Error calculus and regularity of Poisson functionals: the lent particle method

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Abstract

We propose a new method to apply the Lipschitz functional calculus of local Dirichlet forms to Poisson random measures. ***To cite this article: N. Bouleau, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Calcul d'erreur et régularité des fonctionnelles de Poisson : la méthode de la particule prêtée. Nous proposons une nouvelle méthode pour appliquer le calcul fonctionnel lipschitzien des formes de Dirichlet locales aux mesures aléatoires de Poisson. ***Pour citer cet article : N. Bouleau, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Notation and basic formulae

Let us consider a local Dirichlet structure with carré du champ $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ where $(X, \mathcal{X}, \nu)$ is a σ-finite measured space called *bottom-space*. Singletons are in $\mathcal{X}$ and ν is diffuse, $\mathbf{d}$ is the domain of the Dirichlet form $\epsilon[u] = 1/2 \int \gamma[u]\,\mathrm{d}\nu$. We denote $(a, \mathcal{D}(a))$ the generator in $L^2(\nu)$ (cf. [3]).

A random Poisson measure associated to $(X, \mathcal{X}, \nu)$ is denoted N. Ω is the configuration space of countable sums of Dirac masses on X and $\mathcal{A}$ is the σ-field generated by N, of law $\mathbb{P}$ on Ω. The space $(\Omega, \mathcal{A}, \mathbb{P})$ is called *the up-space*. We write $N(f)$ for $\int f\,\mathrm{d}N$. If $p \in [1, \infty[$ the set $\{\mathrm{e}^{\mathrm{i}N(f)}\colon\ f \text{ real},\ f \in L^1 \cap L^2(\nu)\}$ is total in $L^p_{\mathbb{C}}(\Omega, \mathcal{A}, \mathbb{P})$. We put $\tilde{N} = N - \nu$. The relation $\mathbb{E}(\tilde{N} f)^2 = \int f^2\,\mathrm{d}\nu$ extends and gives sense to $\tilde{N}(f)$, $f \in L^2(\nu)$. The Laplace functional and the differential calculus with γ yield

$$\forall f \in \mathbf{d},\ \forall h \in \mathcal{D}(a) \quad \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\tilde{N}(f)}\left(\tilde{N}\big(a[h]\big) + \frac{\mathrm{i}}{2} N\big(\gamma[f, h]\big)\right)\right] = 0. \tag{1}$$

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2. Product, particle by particle, of a Poisson random measure by a probability measure

Given a probability space $(R, \mathcal{R}, \rho)$, let us consider a Poisson random measure $N \odot \rho$ on $(X \times R, \mathcal{X} \times \mathcal{R})$ with intensity $\nu \times \rho$ such that for $f \in L^1(\nu)$ and $g \in L^1(\rho)$ if $N(f) = \sum f(x_n)$ then $(N \odot \rho)(fg) = \sum f(x_n) g(r_n)$ where the r_n's are i.i.d. independent of N with law ρ. Calling $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})$ the product of all the factors $(R, \mathcal{R}, \rho)$ involved in the construction of $N \odot \rho$, we obtain the following properties: For an $\mathcal{A} \times \mathcal{X} \times \mathcal{R}$-measurable and positive function F, $\hat{\mathbb{E}} \int F(\omega, x, r) N \odot \rho(\mathrm{d}x\, \mathrm{d}r) = \int F\, \mathrm{d}\rho\, \mathrm{d}N$ $\mathbb{P}$-a.s.

Let us denote by $\mathbb{P}_N$ the measure $\mathbb{P}(\mathrm{d}\omega) N_\omega(\mathrm{d}x)$ on $(\Omega \times X, \mathcal{A} \times \mathcal{X})$. We have the following:

Lemma 2.1. *Let F be $\mathcal{A} \times \mathcal{X} \times \mathcal{R}$-measurable, $F \in \mathcal{L}^2(\mathbb{P}_N \times \rho)$ and such that $\int F(\omega, x, r) \rho(\mathrm{d}r) = 0$ $\mathbb{P}_N$-a.s., then $\int F\, \mathrm{d}(N \odot \rho)$ is well defined, belongs to $L^2(\mathbb{P} \times \hat{\mathbb{P}})$ and*

$$\hat{\mathbb{E}}\left(\int F\, \mathrm{d}(N \odot \rho)\right)^2 = \int F^2\, \mathrm{d}N\, \mathrm{d}\rho \quad \mathbb{P}\text{-}a.s. \tag{2}$$

The argument consists in considering F_n satisfying $\mathbb{E} \int F_n^2\, \mathrm{d}\nu\, \mathrm{d}\rho < +\infty$ and $\mathbb{E} \int (\int |F_n|\, \mathrm{d}\nu)^2\, \mathrm{d}\rho < +\infty$ and using the relation $\hat{\mathbb{E}}(\int F_n\, \mathrm{d}(N \odot \rho))^2 = (\int F_n\, \mathrm{d}\rho dN)^2 - \int (\int F_n\, \mathrm{d}\rho)^2\, \mathrm{d}N + \int F_n^2\, \mathrm{d}\rho\, \mathrm{d}N$ $\mathbb{P}$-a.s.

3. Construction by Friedrichs' method and expression of the gradient

(a) We suppose the space by $\mathbf{d}$ of the bottom structure is separable, then a gradient exists (cf. [3] Chap. V, p. 225 et seq.). We denote it $\flat$ and choose it with values in the space $L^2(R, \mathcal{R}, \rho)$. Thus, for $u \in \mathbf{d}$ we have $u^\flat \in L^2(\nu \times \rho)$, $\gamma[u] = \int (u^\flat)^2\, \mathrm{d}\rho$ and $\flat$ satisfies the chain rule. We suppose in addition, what is always possible, that $\flat$ takes its values in the subspace orthogonal to the constant 1, i.e.

$$\forall u \in \mathbf{d} \quad \int u^\flat\, \mathrm{d}\rho = 0 \quad \nu\text{-a.s.} \tag{3}$$

This hypothesis is important here as in many applications (cf. [2] Chap. V §4.6). We suppose also, but this is not essential (cf. [3] p. 44) $1 \in \mathbf{d}_{\text{loc}}$ $\gamma[1] = 0$ so that $1^\flat = 0$.

(b) We define a pre-domain $\mathcal{D}_0$ dense in $L^2_{\mathbb{C}}(\mathbb{P})$ by

$$\mathcal{D}_0 = \left\{ \sum_{p=1}^{m} \lambda_p\, \mathrm{e}^{\mathrm{i}\tilde{N}(f_p)};\ m \in \mathbb{N}^*, \lambda_p \in \mathbb{C}, f_p \in \mathcal{D}(a) \cap L^1(\nu) \right\}.$$

(c) We introduce the creation operator inspired from quantum mechanics (see [7–9,1,5,6] and [10] among others) defined as follows

$$\varepsilon_x^+(\omega) \text{ equals } \omega \text{ if } x \in \operatorname{supp}(\omega), \quad \text{and} \quad \text{equals } \omega + \varepsilon_x \text{ if } x \notin \operatorname{supp}(\omega) \tag{4}$$

so that

$$\varepsilon_x^+(\omega) = \omega \quad N_\omega\text{-a.e. } x \quad \text{and} \quad \varepsilon_x^+(\omega) = \omega + \varepsilon_x \quad \nu\text{-a.e. } x. \tag{5}$$

This map is measurable and the Laplace functional shows that for an $\mathcal{A} \times \mathcal{X}$-measurable $H \geqslant 0$,

$$\mathbb{E} \int \varepsilon^+ H\, \mathrm{d}\nu = \mathbb{E} \int H\, \mathrm{d}N. \tag{6}$$

Let us remark also that by (5), for $F \in \mathcal{L}^2(\mathbb{P}_N \times \rho)$

$$\int \varepsilon^+ F\, \mathrm{d}(N \odot \rho) = \int F\, \mathrm{d}(N \odot \rho) \quad \mathbb{P} \times \hat{\mathbb{P}}\text{-a.s.} \tag{7}$$

(d) We defined a gradient $\sharp$ for the up-structure on $\mathcal{D}_0$ by putting for $F \in \mathcal{D}_0$

$$F^\sharp = \int (\varepsilon^+ F)^\flat\, \mathrm{d}(N \odot \rho) \tag{8}$$

this definition being justified by the fact that for $\mathbb{P}$-a.e. ω the map $y \mapsto F(\varepsilon_y^+(\omega)) - F(\omega)$ is in $\mathbf{d}$, $\varepsilon^+ F$ belongs to $L^\infty(\mathbb{P}) \otimes \mathbf{d}$ algebraic tensor product, and $(\varepsilon^+ F - F)^\flat = (\varepsilon^+ F)^\flat \in L^2(\mathbb{P}_N \times \rho)$.

For $F, G \in \mathcal{D}_0$ of the form

$$F = \sum_p \lambda_p \, \mathrm{e}^{\mathrm{i}\tilde{N}(f_p)} = \Phi\big(\tilde{N}(f_1), \dots, \tilde{N}(f_m)\big), \qquad G = \sum_q \mu_q \, \mathrm{e}^{\mathrm{i}\tilde{N}(g_q)} = \Psi\big(\tilde{N}(g_1), \dots, \tilde{N}(g_n)\big)$$

we compute using (2), (3) and (7) (in the spirit of Prop. 1 of [9] or Lemma 1.2 of [6])

$$\hat{\mathbb{E}}\big[F^\sharp \overline{G^\sharp}\big] = \sum_{p,q} \lambda_p \overline{\mu_q} \, \mathrm{e}^{\mathrm{i}\tilde{N}(f_p) - \mathrm{i}\tilde{N}(g_q)} N\big(\gamma[f_p, g_q]\big) \tag{9}$$

and we have:

Proposition 3.1. *If we put* $A_0[F] = \sum_p \lambda_p \, \mathrm{e}^{\mathrm{i}\tilde{N}(f_p)}(\mathrm{i}\tilde{N}(a[f_p]) - \frac{1}{2}N(\gamma[f_p]))$ *it comes*

$$\mathbb{E}\big[A_0[F]\overline{G}\big] = -\frac{1}{2}\mathbb{E}\sum_{p,q} \Phi'_p \overline{\Psi'_q} N\big(\gamma[f_p, g_q]\big). \tag{10}$$

In order to show that $A_0[F]$ does not depend on the form of F, by (10) it is enough to show that the expression $\sum_{p,q} \Phi'_p \overline{\Psi'_q} N(\gamma[f_p, g_q])$ depends only on F and G. But this comes from (9) since $F^\sharp$ and $G^\sharp$ depend only on F and G.

By this proposition, A_0 is symmetric on $\mathcal{D}_0$, negative, and the argument of Friedrichs applies (cf. [3] p. 4), A_0 extends uniquely to a selfadjoint operator $(A, \mathcal{D}(A))$ which defines a closed positive (hermitian) quadratic form $\mathcal{E}[F] = -\mathbb{E}[A[F]\overline{F}]$. By (10) contractions operate and (cf. [3]) $\mathcal{E}$ is a Dirichlet form which is local with carré du champ denoted Γ and the up-structure obtained $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ satisfies

$$\forall f \in \mathbf{d}, \quad \tilde{N}(f) \in \mathbb{D} \quad \text{and} \quad \Gamma\big[\tilde{N}(f)\big] = N\big(\gamma[f]\big). \tag{11}$$

The operator $\sharp$ extends to a gradient for Γ as a closed operator from $L^2(\mathbb{P})$ into $L^2(\mathbb{P} \times \hat{\mathbb{P}})$ with domain $\mathbb{D}$ which satisfies the chain rule and may be computed on functionals $\Phi(\tilde{N}(f_1), \dots, \tilde{N}(f_m))$, Φ Lipschitz and $\mathcal{C}^1$ and their limits in $\mathbb{D}$ (as done in [4]).

Formula (8) for $\sharp$ can be extended from $\mathcal{D}_0$ to $\mathbb{D}$. Let us introduce the space $\underline{\mathbb{D}}$ closure of $\mathcal{D}_0 \otimes \mathbf{d}$ for the norm

$$\|H\|_{\underline{\mathbb{D}}} = \left(\mathbb{E}\int \gamma\big[H(\omega, \cdot)\big](x) N(\mathrm{d}x)\right)^{1/2} + \mathbb{E}\int \big|H(\omega, x)\big| \xi(x) N(\mathrm{d}x)$$

where $\xi > 0$ is a fixed function such that $N(\xi) \in L^2(\mathbb{P})$.

Theorem 3.2. *The formula* $F^\sharp = \int (\varepsilon^+ F)^\flat \, \mathrm{d}(N \odot \rho)$ *decomposes as follows*

$$F \in \mathbb{D} \xmapsto{\varepsilon^+} \varepsilon^+ F \in \underline{\mathbb{D}} \xmapsto{\flat} (\varepsilon^+ F)^\flat \in L^2_0(\mathbb{P}_N \times \rho) \overset{\mathrm{d}(N \odot \rho)}{\longmapsto} F^\sharp \in L^2(\mathbb{P} \times \hat{\mathbb{P}})$$

where each operator is continuous on the range of the preceding one, $L^2_0(\mathbb{P}_N \times \rho)$ *denoting the closed subspace of* $L^2(\mathbb{P}_N \times \rho)$ *of* ρ*-centered elements, and we have*

$$\Gamma[F] = \hat{\mathbb{E}}|F^\sharp|^2 = \int \gamma[\varepsilon^+ F] \, \mathrm{d}N. \tag{12}$$

4. The lent particle method

Let us consider, for instance, a real process Y_t with independent increments and Lévy measure σ integrating x^2, Y_t being supposed centered without Gaussian part. We assume that σ has an l.s.c. density so that a local Dirichlet structure may be constructed on $\mathbb{R}\backslash\{0\}$ with carré du champ $\gamma[f] = x^2 f'^2(x)$. If N is the random Poisson measure with intensity $\mathrm{d}t \times \sigma$ we have $\int_0^t h(s)\,\mathrm{d}Y_s = \int 1_{[0,t]}(s) h(s) x \tilde{N}(\mathrm{d}s\,\mathrm{d}x)$ and the choice done for γ gives $\Gamma[\int_0^t h(s)\,\mathrm{d}Y_s] = \int_0^t h^2(s)\,\mathrm{d}[Y, Y]_s$ for $h \in L^2_{\mathrm{loc}}(\mathrm{d}t)$. In order to study the regularity of the random variable $V = \int_0^t \varphi(Y_{s-})\,\mathrm{d}Y_s$ where φ is Lipschitz and $\mathcal{C}^1$, we have two ways:

(a) We may represent the gradient $\sharp$ as $Y_t^\sharp = B_{[Y,Y]_t}$ where B is a standard auxiliary independent Brownian motion. Then by the chain rule $V^\sharp = \int_0^t \varphi'(Y_{s-})(Y_{s-})^\sharp \,\mathrm{d}Y_s + \int_0^t \varphi(Y_{s-})\,\mathrm{d}B_{[Y]_s}$ now, using $(Y_{s-})^\sharp = (Y_s^\sharp)_-$, a classical but rather tedious stochastic computation yields

$$\Gamma[V] = \hat{\mathbb{E}}\big[V^{\sharp 2}\big] = \sum_{\alpha \leqslant t} \Delta Y_\alpha^2 \left(\int_{]\alpha}^{t} \varphi'(Y_{s-})\,\mathrm{d}Y_s + \varphi(Y_{\alpha-}) \right)^2. \tag{13}$$

Since V has real values the *energy image density property* holds, and V has a density as soon as $\Gamma[V]$ is strictly positive a.s. what may be discussed using the relation (13).

(b) Another more direct way consists in applying the theorem. For this we define $\flat$ by choosing η such that $\int_0^1 \eta(r)\,\mathrm{d}r = 0$ and $\int_0^1 \eta^2(r)\,\mathrm{d}r = 1$ and putting $f^\flat = xf'(x)\eta(r)$.

1°. First step. We add a particle (α, x) i.e. a jump to Y at time α with size x what gives $\varepsilon^+ V - V = \varphi(Y_{\alpha-})x + \int_{]\alpha}^t (\varphi(Y_{s-} + x) - \varphi(Y_{s-}))\,\mathrm{d}Y_s$

2°. $V^\flat = 0$ since V does not depend on x, and $(\varepsilon^+ V)^\flat = (\varphi(Y_{\alpha-})x + \int_{]\alpha}^t \varphi'(Y_{s-} + x)x\,\mathrm{d}Y_s)\eta(r)$ because $x^\flat = x\eta(r)$.

3°. We compute $\gamma[\varepsilon^+ V] = \int (\varepsilon^+ V)^{\flat 2}\,\mathrm{d}r = (\varphi(Y_{\alpha-})x + \int_{]\alpha}^t \varphi'(Y_{s-} + x)x\,\mathrm{d}Y_s)^2$.

4°. We take back the particle we gave, because in order to compute $\int \gamma[\varepsilon^+ V]\,\mathrm{d}N$ the integral in N confuses $\varepsilon^+\omega$ and ω. That gives $\int \gamma[\varepsilon^+ V]\,\mathrm{d}N = \int (\varphi(Y_{\alpha-}) + \int_{]\alpha}^t \varphi'(Y_{s-})\,\mathrm{d}Y_s)^2 x^2 N(\mathrm{d}\alpha\,\mathrm{d}x)$ and (13).

We remark that both operators $F \mapsto \varepsilon^+ F$, $F \mapsto (\varepsilon^+ F)^\flat$ are non-local, but instead $F \mapsto \int (\varepsilon^+ F)^\flat\,\mathrm{d}(N \odot \rho)$ and $F \mapsto \int \gamma[\varepsilon^+ F]\,\mathrm{d}N$ are local: taking back the lent particle gives the locality.

References

[1] S. Albeverio, Yu.G. Kondratiev, M. Röckner, Analysis and geometry on configuration spaces, J. Funct. Anal. 154 (2) (1998) 444–500.
[2] N. Bouleau, Error Calculus for Finance and Physics: The Language of Dirichlet Forms, De Gruyter, 2003.
[3] N. Bouleau, F. Hirsch, Dirichlet Forms and Analysis on Wiener Space, De Gruyter, 1991.
[4] L. Denis, A criterion of density for solutions of Poisson-driven SDEs, Probab. Theory Relat. Fields 118 (2000) 406–426.
[5] Y. Ishikawa, H. Kunita, Malliavin calculus on the Wiener–Poisson space and its application to canonical SDE with jumps, Stochastic Process. Appl. 116 (2006) 1743–1769.
[6] Z.-M. Ma, M. Röckner, Construction of diffusions on configuration spaces, Osaka J. Math. 37 (2000) 273–314.
[7] D. Nualart, J. Vives, Anticipative calculus for the Poisson process based on the Fock space, in: Sém. Probabilités XXIV, in: Lecture Notes in Math., vol. 1426, Springer, 1990, pp. 154–165.
[8] J. Picard, On the existence of smooth densities for jump processes, Probab. Theory Relat. Fields 105 (1996) 481–511.
[9] N. Privault, A pointwise equivalence of gradients on configuration spaces, C. R. Acad. Sci. Paris 327 (1998) 677–682.
[10] J.-L. Solé, F. Utzet, J. Vives, Canonical Lévy processes and Malliavin calculus, Stochastic Process. Appl. 117 (2007) 165–187.