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Partial Differential Equations

Inverse problem for the Schrödinger operator in an unbounded strip

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Abstract

We prove an adapted global Carleman estimate and an energy estimate for the Schrödinger operator $H := \mathrm{i}\partial_t + \nabla \cdot (c\nabla)$ in an unbounded strip. Using these estimates, we give a stability result for the diffusion coefficient $c(x, y)$ from the measurement of the normal derivative of the solution on a part of the boundary. ***To cite this article: L. Cardoulis et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Un problème inverse pour l'opérateur de Schrödinger dans une bande. Nous démontrons une estimation globale de Carleman et une estimation d'énergie pour l'opérateur de Schrödinger $H := \mathrm{i}\partial_t + \nabla \cdot (c\nabla)$ dans une bande non bornée. Ces estimations nous permettent de donner un résultat de stabilité pour le coefficient de diffusion $c(x, y)$ à partir de la mesure de la dérivée normale de la solution sur une partie du bord. ***Pour citer cet article : L. Cardoulis et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

L'objet de cette Note est d'obtenir un résultat de stabilité pour le coefficient de diffusion c pour l'opérateur de Schrödinger $H := \mathrm{i}\partial_t + \nabla \cdot (c\nabla)$ dans une bande non bornée de $\mathbb{R}^2$. Nous démontrons tout d'abord une estimation globale de Carleman pour l'opérateur H avec une seule observation sur une partie du bord (cf. Théorème 2.3). Nous prouvons ensuite une inégalité de type Poincaré avec poids (cf. Lemmes 3.5, 3.6). Cette inégalité appliquée à la condition initiale, plus une estimation d'énergie (cf. Lemme 3.2, Théorème 3.3), obtenue en utilisant l'estimation de Carleman, nous permettent d'établir un résultat de stabilité pour le coefficient de diffusion (cf. Théorème 3.7).

1. Introduction

Let $\Omega = \mathbb{R} \times (-\frac{d}{2}, \frac{d}{2})$ be an unbounded strip of $\mathbb{R}^2$ with a fixed width d. We will consider the Schrödinger equation

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$$\begin{cases} Hq := \mathrm{i}\partial_t q + \nabla\cdot(c(x,y)\nabla q) = 0 & \text{in } Q = \Omega\times(0,T),\\ q(x,y,t) = b(x,y,t) & \text{on } \Sigma = \partial\Omega\times(0,T),\\ q(x,y,0) = q_0(x,y) & \text{on } \Omega, \end{cases} \tag{1}$$

where $c(x,y)\in\mathcal{C}^3(\bar\Omega)$ and $c(x,y)\geqslant c_{\min}>0$. Moreover, we assume that c and all its derivatives up to order three are bounded. If we assume that q_0 belongs to $H^4(\Omega)$ and b is sufficiently regular (e.g. $b\in H^1(0,T,H^{\frac{9}{2}+\varepsilon}(\partial\Omega))\cap H^2(0,T,H^{\frac{5}{2}+\varepsilon}(\partial\Omega))$ and some additional conditions), then (1) admits a solution in $H^1(0,T,H^{\frac{3}{2}+\varepsilon}(\Omega))$. We will use this regularity result later. The aim of this Note is to give a stability and uniqueness result for the coefficient $c(x,y)$ using global Carleman estimates and energy estimates. We denote by ν the outward unit normal to Ω on $\Gamma=\partial\Omega$. We denote $\Gamma=\Gamma^+\cup\Gamma^-$, where $\Gamma^+=\{(x,y)\in\Gamma;\ y=\frac{d}{2}\}$ and $\Gamma^-=\{(x,y)\in\Gamma;\ y=-\frac{d}{2}\}$. We use the following notations:

$$\nabla\cdot(c\nabla u)=\partial_x(c\partial_x u)+\partial_y(c\partial_y u),\quad \nabla u\cdot\nabla v=\partial_x u\partial_x v+\partial_y u\partial_y v,\quad \partial_\nu u=\nabla u\cdot\nu,$$

and

$$Q=\Omega\times(0,T),\quad \tilde Q=\Omega\times(-T,T),\quad \Sigma=\Gamma\times(0,T)\quad\text{and}\quad \tilde\Sigma=\Gamma\times(-T,T).$$

Our problem can be stated as follows: Is it possible to determine the coefficient $c(x,y)$ from the measurement of $\partial_\nu(\partial_t q)$ on Γ^+?

Let q (resp. $\tilde q$) be a solution of (1) associated with (c,b,q_0) (resp. $(\tilde c,b,q_0)$) satisfying some regularity properties:

- $\partial_t\tilde q$, $\nabla(\partial_t\tilde q)$ and $\Delta(\partial_t\tilde q)$ are bounded,
- q_0 is a real valued function in $\mathcal{C}^3(\Omega)$,
- q_0 and all its derivatives up to order three are bounded.

Our main result is

$$|c-\tilde c|^2_{H^1(\Omega)}\leqslant C\left|\partial_\nu(\partial_t q)-\partial_\nu(\partial_t\tilde q)\right|^2_{L^2((-T,T)\times\Gamma^+)},$$

where C is a positive constant which depends on (Ω,Γ,T) and where the above norms are weighted Sobolev norms.

The major novelty of this paper is to give an H^1 stability estimate for the diffusion coefficient with only one observation in an unbounded domain.

We prove an adapted global Carleman estimate and an energy estimate for the operator H with a boundary term on Γ^+. Such energy estimate has been proved in [7] for the Schrödinger operator in a bounded domain in order to obtain a controllability result. Then using these estimates and following the method developed by Immanuvilov, Isakov and Yamamoto for the Lamé system in [2,3], we give a stability and uniqueness result for the diffusion coefficient $c(x,y)$. Note that this stability result corresponds to a stability result for two linked coefficients (c and ∇c) with only one observation. For two independent coefficients, in our knowledge, there is no stability result with one observation.

This paper is organized as follows. In Section 2, we give an adapted global Carleman estimate for the operator H. In Section 3, we prove an energy estimate and we give a stability result for the diffusion coefficient c.

2. Global Carleman estimate

Let $c=c(x,y)$ be a bounded positive function in $\mathcal{C}^3(\bar\Omega)$ such that

Assumption 2.1. $c(x,y)\geqslant c_{\min}>0$, c and all its derivatives up to order three are bounded by a positive constant $\tilde C_0$.

Let $q=q(x,y,t)$ be a function equals to zero on $\partial\Omega\times(-T,T)$ and solution of the Schrödinger equation

$$\mathrm{i}\partial_t q+\nabla\cdot\big(c(x,y)\nabla q\big)=f.$$

We prove here a global Carleman-type estimate for q with a single observation acting on a part Γ^+ of the boundary Γ in the right-hand side of the estimate. Let $\tilde\beta$ be a $\mathcal{C}^4(\bar\Omega)$ positive function such that there exists positive constants C_0, C_1, C_{pc} which satisfy

Assumption 2.2.

- $|\nabla\tilde{\beta}| \geqslant C_0 > 0$ in Ω, $\partial_\nu\tilde{\beta} \leqslant 0$ on Γ^-,
- $\tilde{\beta}$ and all its derivatives up to order four are bounded in $\bar{\Omega}$ by C_1,
- $2\Re(D^2\tilde{\beta}(\zeta,\bar{\zeta})) - c\nabla c\cdot\nabla\tilde{\beta}|\zeta|^2 + 2c^2|\nabla\tilde{\beta}\cdot\zeta|^2 \geqslant C_{pc}|\zeta|^2$, for all $\zeta\in\mathbb{C}$

where

$$D^2\tilde{\beta} = \begin{pmatrix} c\partial_x(c\partial_x\tilde{\beta}) & c\partial_x(c\partial_y\tilde{\beta}) \\ c\partial_y(c\partial_x\tilde{\beta}) & c\partial_y(c\partial_y\tilde{\beta}) \end{pmatrix}.$$

Note that the last assertion of Assumption 2.2 expresses the pseudo-convexity condition for the function β. This assumption imposes restrictive conditions for the choice of the functions $c(x,y)$ in connection with the function $\tilde{\beta}$. Note that there exist functions satisfying such assumptions; indeed, if we consider

$$c(x,y)\in\left\{ f\in C^1(\Omega);\ \exists r_0 \text{ positive constant, } \begin{cases} -f\partial_y f\partial_y\tilde{\beta} \geqslant r_0 > 0, \\ f\ \partial_y f\partial_y\tilde{\beta}((\frac{\partial_x f}{\partial_y f})^2+1) + 2f^2(\partial_{yy}\tilde{\beta} + (\partial_y\tilde{\beta})^2) \geqslant r_0 > 0 \end{cases} \right\}$$

then a function $\tilde{\beta}(x,y)=\tilde{\beta}(y)$ is available (for example, $c(x,y) = (\frac{1}{1+x^2}+1)\mathrm{e}^{-y}$ and $\tilde{\beta}(x,y)=\mathrm{e}^y$).

Similar restrictive conditions have been highlighted for the hyperbolic case in [5,6] and for the Schrödinger operator in [4].

Then, we define $\beta = \tilde{\beta} + K$ with $K = m\|\tilde{\beta}\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t\in(-T,T)$, we define the following weight functions

$$\varphi(x,y,t) = \frac{\mathrm{e}^{\lambda\beta(x,y)}}{(T+t)(T-t)}, \qquad \eta(x,y,t) = \frac{\mathrm{e}^{2\lambda K} - \mathrm{e}^{\lambda\beta(x,y)}}{(T+t)(T-t)}.$$

We set $\psi = \mathrm{e}^{-s\eta}q$, $M\psi = \mathrm{e}^{-s\eta}H(\mathrm{e}^{s\eta}\psi)$ for $s>0$. Let H be the operator defined by

$$Hq := \mathrm{i}\partial_t q + \nabla\cdot\big(c(x,y)\nabla q\big) \quad \text{in } \tilde{Q} = \Omega\times(-T,T), \tag{2}$$

and we introduce the following operators following [1]

$$M_1\psi := \mathrm{i}\partial_t\psi + \nabla\cdot(c\nabla\psi) + s^2c|\nabla\eta|^2\psi, \qquad M_2\psi := \mathrm{i}s\partial_t\eta\psi + 2cs\nabla\eta\cdot\nabla\psi + s\nabla\cdot(c\nabla\eta)\psi. \tag{3}$$

Then

$$\int_{-T}^{T}\!\!\int_\Omega |M\psi|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = \int_{-T}^{T}\!\!\int_\Omega |M_1\psi|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + \int_{-T}^{T}\!\!\int_\Omega |M_2\psi|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + 2\Re\left(\int_{-T}^{T}\!\!\int_\Omega M_1\psi\,\overline{M_2\psi}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\right),$$

where $\bar{z}$ is the conjugate of z, $\Re(z)$ its real part and $\Im(z)$ its imaginary part. We have to compute the previous scalar product. Then the following result holds:

Theorem 2.3. *Let H, M_1, M_2 be the operators defined respectively by* (2), (3). *We assume that Assumptions* 2.1 *and* 2.2 *are satisfied. Then there exist $\lambda_0>0$, $s_0>0$ and a positive constant $C = C(\Omega,\Gamma,T)$ such that, for any $\lambda\geqslant\lambda_0$ and any $s\geqslant s_0$, the next inequality holds*:

$$\begin{aligned} & s^3\lambda^4\int_{-T}^{T}\!\!\int_\Omega \mathrm{e}^{-2s\eta}\varphi^3|q|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + s\lambda\int_{-T}^{T}\!\!\int_\Omega \mathrm{e}^{-2s\eta}\varphi|\nabla q|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + \big\|M_1(\mathrm{e}^{-s\eta}q)\big\|^2_{L^2(\tilde{Q})} \\ & \quad + \big\|M_2(\mathrm{e}^{-s\eta}q)\big\|^2_{L^2(\tilde{Q})} \leqslant C\left[s\lambda\int_{-T}^{T}\!\!\int_{\Gamma^+} \mathrm{e}^{-2s\eta}\varphi|\partial_\nu q|^2\partial_\nu\beta\,\mathrm{d}\sigma\,\mathrm{d}t + \int_{-T}^{T}\!\!\int_\Omega \mathrm{e}^{-2s\eta}|Hq|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\right], \end{aligned} \tag{4}$$

for all q satisfying $Hq\in L^2(\Omega\times(-T,T))$, $q\in L^2(-T,T;H_0^1(\Omega))$, $\partial_\nu q\in L^2(-T,T;L^2(\Gamma))$.

3. Inverse problem

In this section, we establish a stability inequality and deduce a uniqueness result for the coefficient c. The Carleman estimate (4) proved in Section 2 will be the key ingredient in the proof of such a stability estimate.

Let q and $\tilde{q}$ be solutions of

$$\begin{cases} \mathrm{i}\partial_t q + \nabla\cdot(c\nabla q) = 0 & \text{in } \Omega\times(0,T),\\ q(x,y,t) = b(x,y,t) & \text{on } \partial\Omega\times(0,T),\\ q(x,y,0) = q_0(x,y) & \text{in } \Omega, \end{cases} \qquad \begin{cases} \mathrm{i}\partial_t \tilde{q} + \nabla\cdot(\tilde{c}\nabla \tilde{q}) = 0 & \text{in } \Omega\times(0,T),\\ \tilde{q}(x,y,t) = b(x,y,t) & \text{on } \partial\Omega\times(0,T),\\ \tilde{q}(x,y,0) = q_0(x,y) & \text{in } \Omega, \end{cases} \tag{5}$$

where c and $\tilde{c}$ both satisfy Assumption 2.1. If we set $u = q - \tilde{q}$, $v = \partial_t u$ and $\gamma = \tilde{c} - c$, then v satisfies

$$\begin{cases} \mathrm{i}\partial_t v + \nabla\cdot(c\nabla v) = \nabla\cdot(\gamma\nabla\partial_t\tilde{q}) = f & \text{in } \Omega\times(0,T),\\ v(x,y,t) = 0 & \text{on } \partial\Omega\times(0,T),\\ v(x,y,0) = \frac{1}{\mathrm{i}}\nabla\cdot(\gamma\nabla q_0) & \text{in } \Omega. \end{cases} \tag{6}$$

We extend the function v on $\Omega\times(-T,T)$ by the formula $v(x,y,t) = -\bar{v}(x,y,-t)$ for every $(x,y,t)\in\Omega\times(-T,0)$.

3.1. Energy estimate

We assume throughout this section that $\gamma\in H_0^1(\Omega)$. We introduce

$$\mathbb{E}(t) = \int_\Omega \mathrm{e}^{-2s\eta(t)}\left|\partial_t u(t)\right|^2 \mathrm{d}x\,\mathrm{d}y + \int_\Omega \varphi^{-1}(t)\mathrm{e}^{-2s\eta(t)}\left|\partial_t\nabla u(t)\right|^2 \mathrm{d}x\,\mathrm{d}y.$$

In this section, we will give an estimation of $\mathbb{E}(0)$.

First Step: We first give an estimation of $\int_\Omega \mathrm{e}^{-2s\eta(0)}|\partial_t u(0)|^2\,\mathrm{d}x\,\mathrm{d}y$.

We set $\psi = e^{-s\eta}v$. With the operator $M_1\psi = \mathrm{i}\partial_t\psi + \nabla\cdot(c\nabla\psi) + s^2|\nabla\eta|^2\psi$, we introduce, following [1], $\mathcal{I} = 2\Im(\int_{-T}^0\int_\Omega M_1\psi\bar{\psi}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t)$.

Assumption 3.1. $\partial_t\tilde{q},\ \nabla(\partial_t\tilde{q}),\ \Delta(\partial_t\tilde{q})$ are bounded by a positive constant.

We have the following estimate:

Lemma 3.2. *We assume that Assumption* 3.1 *is satisfied. Then there exists a positive constant $C = C(\Omega,\Gamma,T)$ such that for any $\lambda\geqslant\lambda_0$ and $s\geqslant s_0$, we have*

$$\mathcal{I} = \int_\Omega \mathrm{e}^{-2s\eta(x,y,0)}\left|\partial_t u(x,y,0)\right|^2 \mathrm{d}x\,\mathrm{d}y$$

$$|\mathcal{I}| \leqslant Cs^{-3/2}\lambda^{-2}\int_\Omega \mathrm{e}^{-2s\eta(x,y,0)}\left(|\gamma|^2 + |\nabla\gamma|^2\right)\mathrm{d}x\,\mathrm{d}y + Cs^{-1/2}\lambda^{-1}\int_{-T}^{T}\int_{\Gamma^+}\mathrm{e}^{-2s\eta}\varphi\partial_\nu\beta|\partial_\nu v|^2\,\mathrm{d}\sigma\,\mathrm{d}t. \tag{7}$$

Second Step: We then give an estimate of $\int_\Omega \varphi^{-1}(0)\mathrm{e}^{-2s\eta(0)}|\partial_t\nabla u(0)|^2\,\mathrm{d}x\,\mathrm{d}y$.

We denote

$$E(t) := \int_\Omega c\varphi^{-1}(t)\mathrm{e}^{-2s\eta(t)}\left|\nabla v(t)\right|^2 \mathrm{d}x\,\mathrm{d}y,$$

where $\varphi^{-1} = \frac{1}{\varphi}$. We give an estimate for $E(0)$ in Theorem 3.3.

Theorem 3.3. *Let v be solution of* (6) *in the following class*

$$v\in C\big([0,T],H^1(\Omega)\big),\qquad \partial_\nu v\in L^2\big(0,T,L^2(\Gamma)\big).$$

We assume that Assumptions 2.1 *and* 2.2 *are checked. Then there exists a positive constant* $C = C(\Omega, \Gamma, T) > 0$ *such that for* s *and* λ *sufficiently large*

$$E(0) \leqslant C\left[s^2\lambda^2 \int_{-T}^{T}\int_{\Gamma^+} \mathrm{e}^{-2s\eta}\varphi\partial_\nu\beta|\partial_\nu v|^2\,\mathrm{d}\sigma\,\mathrm{d}t + s\lambda\iint_Q \mathrm{e}^{-2s\eta}|f|^2\right]. \tag{8}$$

3.2. Stability estimate

Now, following an idea developed in [2] for Lamé system in bounded domains, we give an underestimate for $\mathbb{E}(0)$. We adapt the proof of Lemma 3.2 of [2] to an unbounded domain.

Assumption 3.4.

- q_0 is a real valued function in $\mathcal{C}^3(\Omega)$;
- q_0 and all its derivatives up to order three are bounded;
- $|\nabla\beta\cdot\nabla q_0| \geqslant C > 0$ on Ω.

Lemma 3.5. *We consider the first order partial differential operator*

$$(P_0 g)(x,y) = \partial_x q_0(x,y)\partial_x g(x,y) + \partial_y(x,y)\partial_y g(x,y), \quad P_0 g := \nabla q_0\cdot\nabla g$$

where q_0 *satisfies Assumption* 3.4. *Then there exist positive constants* $\lambda_1 > 0$, $s_1 > 0$ *and* $C = C(\Omega, \Gamma, T)$ *such that for all* $\lambda \geqslant \lambda_1$ *and* $s \geqslant s_1$

$$s^2\lambda^2\int_\Omega \varphi_0\mathrm{e}^{-2s\eta_0}|g|^2\,\mathrm{d}x\,\mathrm{d}y \leqslant C\int_\Omega \varphi_0^{-1}\mathrm{e}^{-2s\eta_0}\,|P_0 g|^2\,\mathrm{d}x\,\mathrm{d}y$$

with $\eta_0(x,y) := \eta(x,y,0)$, $\varphi_0(x,y) := \varphi(x,y,0)$ *and for* $g \in H_0^1(\Omega)$.

Then, we apply Lemma 3.5 to the first order partial differential equations satisfied by γ (resp. $\nabla\gamma$) given by the initial condition in (6) (resp. the derivative of the initial condition in (6)) and we deduce the following result:

Lemma 3.6. *Let* u *be solution of* (6). *We assume that Assumptions* 2.2, 3.1 *and* 3.4 *are satisfied. Then there exists a positive constant* $C = C(\Omega, \Gamma, T)$ *such that for* s *and* λ *sufficiently large, the two following estimates hold true*

$$s^2\lambda^2\int_\Omega \varphi_0\mathrm{e}^{-2s\eta_0}|\gamma|^2\,\mathrm{d}x\,\mathrm{d}y \leqslant C\int_\Omega \varphi_0^{-1}\mathrm{e}^{-2s\eta_0}\left|\partial_t u(x,y,0)\right|^2\mathrm{d}x\,\mathrm{d}y, \tag{9}$$

$$s^2\lambda^2\int_\Omega \mathrm{e}^{-2s\eta_0}|\nabla\gamma|^2\,\mathrm{d}x\,\mathrm{d}y \leqslant C\int_\Omega \varphi_0^{-1}\mathrm{e}^{-2s\eta_0}\left(\left|\nabla\left(\partial_t u(x,y,0)\right)\right|^2 + |\gamma|^2\right)\mathrm{d}x\,\mathrm{d}y, \tag{10}$$

for $\gamma \in H_0^2(\Omega)$.

Theorem 3.7. *Let* q *and* $\tilde q$ *be solutions of* (5) *such that* $c - \tilde c \in H_0^2(\Omega)$. *We assume that Assumptions* 2.1, 2.2, 3.1 *and* 3.4 *are satisfied. Then there exists a positive constant* $C = C(\Omega, \Gamma, T)$ *such that for* s *and* λ *large enough,*

$$\int_\Omega \varphi_0\mathrm{e}^{-2s\eta_0}\left(|c-\tilde c|^2 + |\nabla(c-\tilde c)|^2\right)\mathrm{d}x\,\mathrm{d}y \leqslant C\int_{-T}^{T}\int_{\Gamma^+} \varphi\, e^{-2s\eta}\partial_\nu\beta\left|\partial_\nu(\partial_t q - \partial_t\tilde q)\right|^2\mathrm{d}\sigma\,\mathrm{d}t.$$

Proof. Adding (9) and (10) we obtain (with C a positive constant) using the estimate (7) for $|\mathcal{I}|$ and the energy estimate (8) for $E(0)$

$$s^2\lambda^2 \int_\Omega \varphi_0 \mathrm{e}^{-2s\eta_0}\big(|\nabla\gamma|^2 + |\gamma|^2\big)\,\mathrm{d}x\,\mathrm{d}y \leqslant C \int_\Omega \varphi_0^{-1} \mathrm{e}^{-2s\eta_0}\big(\big|\nabla\big(\partial_t u(x,y,0)\big)\big|^2 + \big|\partial_t u(x,y,0)\big|^2\big)\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant C\big(|\mathcal{I}| + E(0)\big).$$

Then, for s and λ sufficiently large, we obtain the theorem. □

Remark 1. Note that all the previous results proved in $\mathbb{R} \times (-\frac{d}{2}, \frac{d}{2})$ are available in $\mathbb{R}^n \times (-\frac{d}{2}, \frac{d}{2})$ for $n \geqslant 2$ if we adapt the regularity properties of the initial and boundary conditions.

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