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Mathematical Analysis

The Schur–Szegö composition for real polynomials ☆

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Abstract

For two real polynomials in one variable $P=\sum_{j=0}^{n} p_j x^j$, $Q=\sum_{j=0}^{n} q_j x^j$ set $W:=\sum_{j=0}^{n}(p_j q_j/C_{n+k}^{j})x^j$ where $k\in \mathbf{N}\cup 0$. For $k=0$ this is the composition of Schur-Szegö of P and Q. We discuss the question if the numbers of negative, positive and complex roots of P and Q are known, what these numbers can be for W. ***To cite this article: V.P. Kostov, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

La composition de Schur–Szegö de polynômes réels. On définit d'après deux polynômes réels à une variable $P=\sum_{j=0}^{n} p_j x^j$, $Q=\sum_{j=0}^{n} q_j x^j$ le polynôme $W:=\sum_{j=0}^{n}(p_j q_j/C_{n+k}^{j})x^j$ où $k\in \mathbf{N}\cup 0$. Pour $k=0$ on obtient la composition de Schur-Szegö de P et Q Nous discutons la question si le nombre de racines strictement négatives, strictement positives et complexes de P et Q sont connus, quels peuvent être ces nombres pour W. ***Pour citer cet article : V.P. Kostov, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Nous considérons des couples de polynômes réels à une variable $P=\sum_{j=0}^{n} p_j x^j$, $Q=\sum_{j=0}^{n} q_j x^j$. Nous posons $W:=P\ {}_{n+k}^{\ *}\ Q=\sum_{j=0}^{n}(p_j q_j/C_{n+k}^{j})x^j$ où dans ce qui suit $k\in \mathbf{N}\cup 0$. Pour $k=0$ on obtient la *composition de Schur–Szegö* de P et Q, pour $k>0$ on obtient cette composition lorsque P et Q sont considérés comme des polynômes de degré $n+k$ avec k premiers coefficients nuls. Nous considérons la question si $P(0)Q(0)\neq 0$ et si on connaît les nombres g, l et 2γ de racines positives, négatives et complexes de P et Q comptées avec multiplicité, quels peuvent être ces nombres pour W. La réponse exhaustive à cette question pour des polynômes *hyperboliques*, c. à d. dont toutes les racines sont réelles, et dont l'un a toutes ses racines du même signe, est donnée dans l'article [3]. Dans ce cas le vecteur multiplicité de W peut être déduit de ceux de P et Q. Dans l'article [2] la réponse est donnée en cas où P et Q sont hyperboliques sans restriction sur les signes de leurs racines. Dans cette Note nous donnons des conditions suffisantes (pour $n=2$ elle sont nécessaires aussi) sur les triplets $(g,l;2\gamma)$ de P, Q et W pour qu'il existe

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des triplets de polynômes P, Q, W avec de tels triplets $(g, l; 2\gamma)$, voir Théorème 0.2 et Proposition 0.3. On peut étendre les résultats au cas $P(0)Q(0) = 0$ à l'aide de Théorème 1.3 et Remarque 4 de [2].

Remarque 1. On a $P(\alpha x) \, {}_{n+k}^{\;*} \, Q(\beta x) = (P \, {}_{n+k}^{\;*} \, Q)(\alpha\beta x)$ (pour tout k et pour tout $\alpha, \beta \in \mathbf{R}^*$).

Définition 0.1. On désigne par $R_{g,l;2\gamma}$ l'ensemble des polynômes réels de degré n ayant g racines < 0, l racines > 0, γ couples conjugués et pas de racine en 0 où $g + l + 2\gamma = n$ $(*)$. On appelle *triplet de triplets admissibles* (*TTA*) un triplet de triplets $(g, l; 2\gamma)$ vérifiant la condition $(*)$. Un TTA est *réalisable pour un k donné* s'il existe un triplet de polynômes P, Q, W à racines distinctes avec ce TTA. On désigne le TTA des polynômes P, Q, W par $\tau := ((g, l; 2\gamma), (r, s; 2\delta), (u, w; 2\eta))$. On présume que

$$g \geqslant l \geqslant s, \quad r \geqslant s, \quad \text{et} \quad \text{si} \quad r > s, \quad \text{alors} \quad g > s. \tag{1}$$

On peut obtenir ces conditions en appliquant Remarque 1 avec $\alpha = -1$ et/ou $\beta = -1$ et en échangeant si nécessaire P avec Q. Comme $W(0) = P(0)Q(0)$,

$$\text{la somme } l + s + w \text{ est paire.} \tag{2}$$

Théorème 0.2. *Pour $n \geqslant 3$ tout TTA vérifiant les conditions suivantes est réalisable pour tout k :*

$$(\tau_1): u \geqslant g - s - 2\delta, \quad (\tau_2): w \geqslant l - s - 2\delta, \quad (\tau_3): \eta \leqslant \gamma + \delta + s, \tag{3}$$

Proposition 0.3.

1) *Pour $n = 2$, $k = 0$ les TTA* $((0, 0; 2), (0, 0; 2), (2, 0; 0))$ *et* $((0, 0; 2), (0, 0; 2), (0, 2; 0))$ *ne sont pas réalisables. Ils sont réalisables pour tout $k \geqslant 1$ respectivement par les triplets de polynômes $P = Q = x^2 + 2x + 17/16$, $W = x^2/C_{2+k}^2 + 4x/C_{2+k}^1 + (17/16)^2$ et $P(-x)$, $Q(x)$, $W(-x)$. Pour $n = 2$ tous les autres TTA vérifiant les conditions* (1), (2) *et* (3) *et triplets de polynômes P, Q, W à racines simples qui les réalisent sont donnés après Proposition* 1.3 *de la version anglaise.*
2) *Pour $n = 2$ les seuls TTA vérifiant les conditions* (1) *et* (2) *et ne vérifiant pas la condition* (3) *sont* $((2, 0; 0), (2, 0; 0), (0, 0; 2))$ *et* $((2, 0; 0), (2, 0; 0), (0, 2; 0))$. *Ils ne sont réalisables pour aucun k.*

Pour $n = 3$ les résultats des articles [3] et [2] et de cette Note donnent pour tous les TTA la réponse à la question s'ils sont réalisables ou pas. Pour $n = 4$ ils ne la donnent pas (mais on la trouve en direct quand-même) pour seulement quatre TTA (dont trois réalisables et un non-réalisable) vérifiant les conditions (1), (2) mais pas la condition (3), voir Remarque 3 de la version anglaise.

1. English version

We consider couples of real polynomials in one variable $P = \sum_{j=0}^{n} p_j x^j$, $Q = \sum_{j=0}^{n} q_j x^j$. We set $W := P \, {}_{n+k}^{\;*} \, Q = \sum_{j=0}^{n} (p_j q_j / C_{n+k}^{j}) x^j$ where in the whole Note $k \in \mathbf{N} \cup 0$. For $k = 0$ this is the *composition of Schur–Szegö* of P and Q, for $k > 0$ it is this composition when P and Q are considered as degree $n + k$ polynomials with k leading coefficients equal to 0. We consider the question when $P(0)Q(0) \neq 0$ and the numbers g, l and 2γ of positive, negative and complex roots of P and Q counting multiplicities are known, what these numbers can be for W. In paper [3] the exhaustive answer to this question is given in the case when $k = 0$; P and Q are *hyperbolic*, i.e. with real roots only, and all roots of one of them have the same sign. (In this case the multiplicity vector of W can be derived from the ones of P and Q.) In paper [2] the answer is given for any k in the case of hyperbolic P and Q with no restriction upon the sign of their roots. In the present Note we give sufficient conditions (for $n = 2$ necessary as well) upon the triples $(g, l; 2\gamma)$ of P, Q and W so that there should exist triples of polynomials P, Q, W with such triples $(g, l; 2\gamma)$, see Theorem 1.2 and Proposition 1.3. To extend the results to the case $P(0)Q(0) = 0$ one can use Theorem 1.3 and Remark 4 of [2].

Remark 1. One has $P(\alpha x) \, {}_{n+k}^{\;*} \, Q(\beta x) = (P \, {}_{n+k}^{\;*} \, Q)(\alpha\beta x)$ (for any k and for any $\alpha, \beta \in \mathbf{R}^*$).

Definition 1.1. Denote by $R_{g,l;2\gamma}$ the set of real degree n polynomials with g negative, l positive, no zero roots and with γ conjugate couples where $g+l+2\gamma=n$ $(*)$. Call a *triple of admissible triples* (*TAT*) a triple of triples $(g,l;2\gamma)$ satisfying condition $(*)$. A TAT is *realizable for a given k* if there exists a triple of polynomials P, Q, W with distinct roots with this TAT. Denote the TAT of the polynomials P, Q, W by $\tau:=((g,l;2\gamma),(r,s;2\delta),(u,w;2\eta))$. As $W(0)=P(0)Q(0)$,

$$\text{the sum } l+s+w \text{ is even.} \tag{4}$$

Using Remark 1 with $\alpha=-1$ and/or $\beta=-1$ and exchanging if necessary P with Q, we assume that

$$g\geqslant l\geqslant s,\quad r\geqslant s,\quad \text{and}\quad \text{if}\quad r>s,\quad \text{then}\quad g>s. \tag{5}$$

Theorem 1.2. *For $n\geqslant 3$ any TAT satisfying the following conditions is realizable for any k:*

$$(\tau_1): u\geqslant g-s-2\delta,\quad (\tau_2): w\geqslant l-s-2\delta,\quad (\tau_3): \eta\leqslant \gamma+\delta+s. \tag{6}$$

Proposition 1.3.

1) *For $n=2$, $k=0$ the TATs* $((0,0;2),(0,0;2),(2,0;0))$ *and* $((0,0;2),(0,0;2),(0,2;0))$ *are not realizable. They are realizable for any $k\geqslant 1$ respectively by the polynomials $P=Q=x^2+2x+17/16$, $W=x^2/C^2_{2+k}+4x/C^1_{2+k}+(17/16)^2$ and $P(-x)$, $Q(x)$, $W(-x)$. For $n=2$ we list below all other TATs satisfying conditions* (4)–(6) *and polynomials P, Q, W with simple roots realizing them.*
2) *For $n=2$ the only TATs satisfying conditions* (4) *and* (5) *and not satisfying condition* (6) *are* $((2,0;0),(2,0;0),(0,0;2))$ *and* $((2,0;0),(2,0;0),(0,2;0))$. *They are not realizable for any k.*

((2,0;0),(2,0;0),(2,0;0))	x^2+3x+2	x^2+3x+2	$x^2/C^2_{2+k}+9x/(2+k)+2$
((1,1;0),(2,0;0),(1,1;0))	x^2-1	x^2+3x+2	$x^2/C^2_{2+k}-2$
((1,1;0),(0,0;2),(1,1;0))	x^2-1	x^2+1	$x^2/C^2_{2+k}-1$
((1,1;0),(1,1;0),(2,0;0) or (0,2;0))	$x^2+x-1/5$	$x^2\pm x-1/5$	$x^2/C^2_{2+k}\pm x/(2+k)+1/25$
((1,1;0),(1,1;0),(0,0;2))	x^2+x-2	x^2+x-2	$x^2/C^2_{2+k}+x/(2+k)+4$
((2,0;0),(0,0;2),(2,0;0) or (0,2;0))	x^2+3x+2	$x^2\pm 2x+17/16$	$x^2/C^2_{2+k}\pm 6x/(2+k)+17/8$
((2,0;0) or (0,0;2),(0,0;2),(0,0;2))	$x^2+(2\pm 2)x+2$	x^2+1	$x^2/C^2_{2+k}+2$

Remark 2. When P and Q are hyperbolic, i.e. $\gamma=\delta=0$, conditions (6) are necessary and sufficient for the realizability of a TAT, see [2]. The example below shows that when at least one of them is not hyperbolic, then some TATs not satisfying conditions (6) might also be realizable. The example is connected with the following fact: for P hyperbolic with all roots positive or negative it is true that if Q is hyperbolic, then $P\ {}^*_n\ Q$ is also hyperbolic, but (for $n\geqslant 4$) it is not true that if Q is only real, then the number of complex couples of zeros of $P\ {}^*_n\ Q$ is not greater than the one of Q. This fact is related to the notions of multiplier and complex zero decreasing sequences, see [1], and of finite multiplier sequences, see [3].

Example 1. For $n=4$ set $P:=(x+1)^2(x+a)^2$, $Q:=(x+1)^2((x+u)^2+v)$, $a>0$, $v>0$, $u\in\mathbf{R}$. Set $\eta=u^2+v$. One has

$$W=\frac{x^4}{C^4_{4+k}}+\frac{(2a+2)(2u+2)x^3}{C^3_{4+k}}+\frac{(1+4a+a^2)(1+4u+\eta)x^2}{C^2_{4+k}}+\frac{(2a+2a^2)(2u+2\eta)x}{C^1_{4+k}}+a^2\eta.$$

Set $x=ay$, $\xi=1+4u+\eta$. Then $W(ay,a,u,v)=a^2(U+ayT)$ where

$$U=\frac{\xi y^2}{C^2_{4+k}}+\frac{4(u+\eta)y}{C^1_{4+k}}+\eta,\quad T=\frac{ay^3}{C^4_{4+k}}+\frac{4(a+1)(u+1)y^2}{C^3_{4+k}}+\frac{(4+a)\xi y}{C^2_{4+k}}+\frac{4(u+\eta)}{C^1_{4+k}}.$$

Suppose first that $u=0$. Fix $\eta=v\in(0,(4+k)/(2+k))$. Hence, U has no real roots. Choose then $a>0$ small enough so that W have no real roots (the choice is possible because $\deg U=2$, $\deg(ayT)=4$ and the leading coefficients of U and ayT are positive). The TAT corresponding to P, Q, W equals $\tau^1=((4,0;0),(2,0;2),(0,0;4))$.

Consider next the case when $u < 0$ is small and $0 < v \ll u^2$. Then U has two distinct positive roots. For $a > 0$ small enough the polynomial $U + ayT$ has only two real roots. They are close to the ones of U (hence distinct and positive). The TAT of P, Q, W in the second case equals $\tau^2 = ((4, 0; 0), (2, 0; 2), (0, 2; 2))$. In both cases one can perturb P and Q to make all roots distinct.

Remark 3. For $n = 3$ one deduces from the results of papers [3] and [2] and of the present Note for every TAT whether it is realizable or not. For $n = 4$ exactly four TATs satisfy conditions (4), (5) and do not satisfy conditions (6): τ^1, τ^2, $\tau^3 = ((4, 0; 0), (2, 0; 2), (0, 4; 0))$ and $\tau^4 = ((2, 2; 0), (2, 0; 2), (0, 0; 4))$. The TAT τ^3 is not realizable – by the Descartes rule there must be respectively four, none, $\geqslant 2$ and $\leqslant 2$ sign changes in the sequence of coefficients of $W(x)$, $P(x)$, $Q(-x)$ and $Q(x)$ which is impossible. To realize τ^4 set $P = Q = (x^2 - 1)^2$, hence $W = gx^4 + hx^2 + 1$ ($g > 0$, $h > 0$) has no real roots; then perturb P and Q to make their roots distinct (real or complex accordingly).

Proof of Proposition 1.3. Set $P := x^2 + ax + b$, $Q := x^2 + cx + d$. If $P, Q \in R_{0,0;2}$, i.e. $a^2 < 4b$, $c^2 < 4d$, then $(ac/2)^2 < 4bd$, i.e. $P \mathbin{{}_{2+0}^{*}} Q \in R_{0,0;2}$. The rest of part 1) is checked directly. Part 2) follows from Proposition 1.2 of [2]. □

Proof of Theorem 1.2. We prove the theorem by induction on n (in some particular cases we construct triples P, Q, W directly). We describe *Procedures A)–M)* which construct triples of degree n polynomials P, Q, W realizing any prescribed TAT when one can construct such triples for $n - 1$ or $n - 2$. The induction base is $n = 2$ (Proposition 1.3), the induction step is {1}: $(n - 1, k + 1) \Rightarrow (n, k)$ (in A), B)) or {2}: $(n - 2, k + 2) \Rightarrow (n, k)$ (in C) – M)). For $n = 3$ we use only step {1}. The case $n = 2$, $k = 0$ (see Proposition 1.3) is never involved. We list the TATs $\tau_A - \tau_M$ whose realizability implies the one of τ:

A)	$\tau_A := ((g - 1, l; 2\gamma),$	$(r - 1, s; 2\delta),$	$(u - 1, w; 2\eta))$
B)	$\tau_B := ((g, l - 1; 2\gamma),$	$(r - 1, s; 2\delta),$	$(u, w - 1; 2\eta))$
C)	$\tau_C := ((g - 2, l; 2\gamma),$	$(r, s; 2(\delta - 1)),$	$(u, w; 2(\eta - 1)))$
D)	$\tau_D := ((g - 2, l; 2\gamma),$	$(r, s; 2(\delta - 1)),$	$(u - 2, w; 2\eta))$
E)	$\tau_E := ((g - 2, l; 2\gamma),$	$(r, s; 2(\delta - 1)),$	$(u, w - 2; 2\eta))$
F)	$\tau_F := ((g, l; 2(\gamma - 1)),$	$(r - 2, s; 2\delta),$	$(u, w; 2(\eta - 1)))$
G)	$\tau_G := ((g - 1, l - 1; 2\gamma),$	$(r, s; 2(\delta - 1)),$	$(u - 1, w - 1; 2\eta))$
H)	$\tau_H := ((g - 1, l - 1; 2\gamma),$	$(r - 1, s - 1; 2\delta),$	$(u, w; 2(\eta - 1)))$
I)	$\tau_I := ((g - 1, l - 1; 2\gamma),$	$(r - 1, s - 1; 2\delta),$	$(u - 2, w; 2\eta))$
J)	$\tau_J := ((g - 1, l - 1; 2\gamma),$	$(r - 1, s - 1; 2\delta),$	$(u, w - 2; 2\eta))$
K)	$\tau_K := ((g, l; 2(\gamma - 1)),$	$(r, s; 2(\delta - 1)),$	$(u - 2, w; 2\eta))$
L)	$\tau_L := ((g, l; 2(\gamma - 1)),$	$(r, s; 2(\delta - 1)),$	$(u, w - 2; 2\eta))$
M)	$\tau_M := ((g, l; 2(\gamma - 1)),$	$(r, s; 2(\delta - 1)),$	$(u, w; 2(\eta - 1)))$

We show first that it is always possible to choose a suitable procedure which carries out the inductive step. After this we describe the procedures. Denote by Λ one of the letters A, B, ..., M and by $(\tau_{\Lambda;j})$, $j = 1, 2, 3$, the analogs of conditions (τ_j) (see (6)) when the TAT τ is replaced by the TAT τ_Λ; (τ_Λ) stands for $(\tau_{\Lambda;j})$, $j = 1, 2, 3$, together. Procedure Λ) is defined if and only if τ_Λ is well defined and satisfies condition (τ_Λ) and the analogs of (4), (5). E.g. Procedure C) is defined if and only if $g \geqslant l + 2$, $\delta \geqslant 1$, $\eta \geqslant 1$ and there hold conditions $(\tau_{C;1}) = (\tau_1)$, $(\tau_{C;3}) = (\tau_3)$ and $(\tau_{C;2})$: $w \geqslant l - s - 2(\delta - 1)$. Notice that each of conditions (τ_A), (τ_B), (τ_F) and (τ_H) is equivalent to condition (6).

If one of Procedures A) or H) can be performed, then the inductive step can be carried out. So we explain how to carry it out when neither A) nor H) is defined. Procedure A) is not defined exactly if one of the following conditions holds: $g = l$ or $r = s$ or $g = s$ or $u = 0$. If $g = l$, then one can consider instead of $P(x)$, $Q(x)$, $W(x)$ the triple $P(-x)$, $Q(x)$, $W(-x)$ and apply Procedure B) instead of Procedure A). If $g = s$, then $l = s$ and one can exchange P with Q, so we assume that Procedure A) is not defined only when $r = s$ or $u = 0$. Procedure H) is not defined only when $s = 0$ or $\eta = 0$. Four cases are possible:

1) $r = s = 0$, 2) $r = s \neq 0$, $\eta = 0$, 3) $u = s = 0 \neq r$, $\eta \neq 0$, 4) $u = \eta = 0$.

In Case 1) n is even, $\delta = n/2$ and one of Procedures C), D), E), G), K), L), M) can be performed. (If $g = l$, one has to modify Procedures C), D), E) by replacing $P(x)$, $Q(x)$ with $P(-x)$, $Q(-x)$.) Conditions (τ_C), (τ_D), (τ_E), (τ_G), (τ_K), (τ_L), (τ_M) are trivially true when $\delta = n/2$.

In Case 2) n is even and one performs Procedure I) when condition $(\tau_{I;1})$ holds. (Condition $(\tau_{I;2})$ coincides with condition (τ_2) and condition $(\tau_{I;3})$ is trivially true for $\eta = 0$.) Condition $(\tau_{I;1})$ fails only when $g - s - 2\delta \leqslant u < g - s - 2\delta + 2$, i.e. $u = g - s - 2\delta + \chi$, $\chi = 0$ or 1. In this case try to perform Procedure J). Conditions (τ_J) hold except when $w = l - s - 2\delta + \psi$, $\psi = 0$ or 1. If both conditions (τ_I) and (τ_J) fail, then $n = u + w = g + l - 2s - 4\delta + \chi + \psi = n - 2s - 4\delta + \chi + \psi$. The right-hand side is $< n$ (which is a contradiction) except for $s = \chi = \psi = 1$, $\delta = 0$. In this case $u = g$, $w = l$ and as $\eta = 0$, one has $\gamma = 0$. Hence P, Q, W are hyperbolic and the TAT τ is realizable by Theorem 1.5 of [2].

In Case 3) perform Procedure F) unless a) $r = 1$ or b) $\gamma = 0$ (conditions (τ_F) coincide with conditions (6)). If a) $r = 1$, perform Procedure B) unless $l = 0$ – conditions (τ_B) coincide with conditions (6), and if $r = 1$, $u = s = 0$, then n is odd and $w \geqslant 1$. But for $r = 1$, $l = s = 0$ by (4) w is even while $w + 2\eta = n$ must be odd – a contradiction. Hence $l = 0$ is impossible. If b) $\gamma = 0$, then we can assume that $\delta > 0$ (for $\gamma = \delta = 0$, the TAT is realizable by Theorem 1.5 of [2]). Perform Procedure B) unless $l = 0$ or $w = 0$. If $\gamma = l = 0$, then perform Procedure C) (conditions (τ_C) are readily checked). If $\gamma = w = u = 0$, then n is even and one can carry out again Procedure C) except when $g = l$ and the first of conditions (5) is not fulfilled for τ_C. In this case one can consider instead of $P(x)$, $Q(x)$, $W(x)$ the polynomials $P(-x)$, $Q(x)$, $W(-x)$. In the modified Procedure C) (called C1)) the TAT τ_{C1} equals $((g, g-2; 0), (r, 0; 2(\delta-1)), (0, 0; 2(\eta-1)))$. Conditions (τ_3) and $(\tau_{C;3})$ coincide, and so do (τ_1) and $(\tau_{C;1}) = (\tau_{C1;2})$: $0 \geqslant g - 2\delta$. Condition $(\tau_{C;2}) = (\tau_{C1;1})$ reads $0 \geqslant g - 2\delta + 2$. It is not fulfilled only for $g = 2\delta$ (by (4) g is even). In this case set $P := (x^2-1)^{2\delta} =: Q$. Hence W is even and with positive coefficients, i.e. W has no real roots. Perturb P and Q so that all roots become distinct, the 2δ-fold positive root of Q (resp. of P) split into δ conjugate couples (resp. into 2δ positive roots) while the 2δ-fold negative roots split into 2δ simple real roots. The perturbed triple realizes the TAT $((g, g; 0), (g, 0; g), (0, 0; 2g))$.

In Case 4), if $s > 0$, perform Procedure J). Condition $(\tau_{J;1})$ coincides with condition (τ_1), conditions $(\tau_{\Lambda;2})$ and $(\tau_{\Lambda;3})$ are trivially true for any Λ when $w = n$, $\eta = 0$. If $s = 0$, then one can perform Procedure B) unless $l = 0$ or $r = 0$ (condition $(\tau_{B;1})$ coincides with condition (τ_1)). If $r = s = 0$, this is Case 1). If $l = s = 0$, then one can perform Procedure E) unless $\delta = 0$ or $g < 2$ (condition $(\tau_{E;1})$ coincides with condition (τ_1)). If $\delta = s = u = l = \eta = 0$, then condition (τ_1) implies $g = 0$ and the TAT is of the form $\Psi := ((0, 0; n), (n, 0; 0), (0, n; 0))$. Condition (4) implies that n is even. Consider polynomials P and Q respectively with $n/2$ distinct double positive roots and with n distinct negative roots. By [3] $P \ {}^*_n\ Q$ has n distinct positive roots. By Proposition 1.2 of [2] W has (for all k) n positive roots. One can deduce from the proof of this proposition that in our case all these roots are distinct. Perturb P so that it have $n/2$ complex conjugate couples. The perturbed triple P, Q, W realizes the TAT Ψ. If $g < 2$, $s = u = l = \eta = 0$, then exchange P with Q – this preserves condition (τ_1) (the quantity $g - s - 2\delta$ changes to $r - l - 2\gamma$ which is equal to it due to $g + 2\gamma = r + 2\delta = n$). After this one can perform Procedure E) unless one has $g < 2$, $r < 2$ or $\delta = 0$ (the case $\delta = 0$ was treated already). If $g < 2$, $r < 2$, $s = u = l = \eta = 0$, then $\gamma > 0$, $\delta > 0$ and one can perform Procedure L) (conditions (τ_L) are checked directly).

We describe in detail only *Procedure C)*, in the other cases we only indicate the differences. Set $P_0 := x^{n-2}(x+1)^2$, $Q_0 := x^{n-2}(x^2+1)$, hence, $P_0 \ {}^*_{n+k}\ Q_0 = x^{n-2}(ax^2+b)$, $a > 0$, $b > 0$. Denote by P_1 and Q_1 two degree $n-2$ monic polynomials with distinct roots, $P_1 \in R_{g-2,l;2\gamma}$, $Q_1 \in R_{r,s;2(\delta-1)}$ such that $P_1 \ {}^*_{n+k}\ Q_1 \in R_{u,w;2(\eta-1)}$, i.e. to the triple of polynomials P_1, Q_1, $P_1 \ {}^*_{n+k}\ Q_1$ there corresponds the TAT τ_C. Consider for $\varepsilon > 0$ small enough the polynomials $P(x,\varepsilon) := \varepsilon^{n-2}P_1(x/\varepsilon)(x+1)^2$ and $Q(x,\varepsilon) := \varepsilon^{n-2}Q_1(x/\varepsilon)(x^2+1)$. They are deformations of P_0, Q_0, i.e. $P(x,0) = P_0(x)$, $Q(x,0) = Q_0(x)$. One has $P \in R_{g,l;2\gamma}$, $Q \in R_{r,s;2\delta}$ and $P \ {}^*_{n+k}\ Q \in R_{u,w;2\eta}$. To check the last inclusion set $x = \varepsilon y$. One has

$$T := \varepsilon^{4-2n} P(\varepsilon y, \varepsilon) \ {}^*_{n+k}\ Q(\varepsilon y, \varepsilon) = \big(P_1(y)(\varepsilon y+1)^2\big) \ {}^*_{n+k}\ \big(Q_1(y)(\varepsilon^2 y^2+1)\big),$$

i.e. T is a deformation of $P_1 \ {}^*_{n+k}\ Q_1$. Hence, $n-2$ of the roots of T are close to the roots of $P_1 \ {}^*_{n+k}\ Q_1$, i.e. T has u negative, w positive and $2(\eta-1)$ complex simple roots "not distant from 0", and two complex conjugate roots "distant from 0", they are close to the roots of $a\varepsilon^2 y^2 + b$. For $\varepsilon > 0$ small enough one has $T \in R_{u,w;2\eta}$ and to construct P and Q one has to fix ε. For the TAT τ (resp. τ_C) one has $s = \min(g, l, r, s)$ (resp. $s = \min(g-2, l, r, s)$), therefore both TATs satisfy conditions (4) and (5). Conditions (τ_C) read:

$$(\tau_{C;1})\colon\ u \geqslant (g-2) - s - 2(\delta-1), \quad (\tau_{C;2})\colon\ w \geqslant l - s - 2(\delta-1), \quad (\tau_{C;3})\colon\ \eta - 1 \leqslant \gamma + (\delta-1) + s.$$

Conditions $(\tau_{C;1})$ and $(\tau_{C;3})$ coincide respectively with (τ_1) and (τ_3) while $(\tau_{C;2})$ implies (τ_2). Procedure C) constructs polynomials P and Q realizing all TATs that satisfy the conditions $g \geqslant l+2$, $\delta \geqslant 1$, $\eta \geqslant 1$ and (τ_C). To define *Procedure F)* exchange P with Q in Procedure C).

In *Procedure A) (resp. B))* set $P_0 := x^{n-1}(x+1)$ (resp. $P_0 := x^{n-1}(x-1)$), $Q_0 := x^{n-1}(x+1)$. Hence, $P_0 \,{}^{*}_{n+k}\, Q_0 = x^{n-1}(ax+b)$ where $a>0$, $b>0$ (resp. $a>0$, $b<0$). The triple P_1, Q_1, $P_1 \,{}^{*}_{n+k}\, Q_1$ realizes the TAT τ_A (resp. τ_B) and $(ax+b)$ adds a negative (resp. positive) root.

In *Procedure D) (resp. E))* set $P_0 := x^{n-2}(x^2+9x+1)$, $Q_0 := x^{n-2}(x^2+x+2)$ (resp. $Q_0 := x^{n-2}(x^2-x+2)$); $P_0 \,{}^{*}_{n+k}\, Q_0$ has two negative (resp. positive) "distant" roots for all k.

In *Procedure G)* set $P_0 := x^{n-2}(x^2-1)$, $Q_0 := x^{n-2}(x^2+1)$. Hence, $P_0 \,{}^{*}_{n+k}\, Q_0$ is of the form $x^{n-2}(ax^2-b)$, $a>0$, $b>0$. One has $P_1 \in R_{g-1,l-1;2\gamma}$, $Q_1 \in R_{r,s;2(\delta-1)}$, $P_1 \,{}^{*}_{n+k}\, Q_1 \in R_{u-1,w-1;2\eta}$. The factor (ax^2-b) adds to $P \,{}^{*}_{n+k}\, Q$ a positive and a negative root, so $P \,{}^{*}_{n+k}\, Q \in R_{u,w;2\eta}$.

In *Procedure H)* set $P_0 := x^{n-2}(x^2-1) =: Q_0$, thus $P_0 \,{}^{*}_{n+k}\, Q_0 = x^{n-2}(ax^2+b)$, $a>0$, $b>0$.

In *Procedure I) (resp. J))* set $R(x) := x^2+3x-1$, $P_0 := x^{n-2}R(x)$, $Q_0 := x^{n-2}R(x)$ (resp. $Q_0 := x^{n-2}R(-x)$). Hence, $P_0 \,{}^{*}_{n+k}\, Q_0 = x^{n-2}S(x)$, resp. $P_0 \,{}^{*}_{n+k}\, Q_0 = x^{n-2}S(-x)$, where $S(x) = (x^2/C^n_{n+k}) + (9x/C^{n-1}_{n+k}) + 1/C^{n-2}_{n+k}$, hence $S(x)$ has two negative roots for all k.

To define *Procedures K), L), M)* set $U(x) := x^2+x+\sigma$ where $\sigma > 1/4$. Set in K), L), M) $P_0 := x^{n-2}U(x)$, set $Q_0 := P_0$ in K), M) and $Q_0 := x^{n-2}U(-x)$ in L). Hence $P_0 \,{}^{*}_{n+k}\, Q_0 = (x^n/C^n_{n+k}) + (x^{n-1}/C^{n-1}_{n+k}) + (x^{n-2}\sigma^2/C^{n-2}_{n+k})$ in K), M); in L) this is $(P_0 \,{}^{*}_{n+k}\, Q_0)(-x)$. There remains to choose σ fulfilling also the condition $(n-1)(k+1) > 4\sigma^2 n(k+2)$ in K), L) or $(n-1)(k+1) < 4\sigma^2 n(k+2)$ in M). $\square$

References

[1] T. Craven, G. Csordas, Composition theorems, multiplier sequences and complex zero decreasing sequences, in: Value distribution theory and related topics, in: Adv. Complex Anal. Appl., vol. 3, Kluwer Acad. Publ., Boston, MA, 2004, pp. 131–166.

[2] V.P. Kostov, The Schur–Szegö composition for hyperbolic polynomials, C. R. Acad. Sci. Paris, Ser. I 345 (9) (2007) 483–488; doi:10.1016/j.crma.2007.10.003.

[3] V.P. Kostov, B.Z. Shapiro, On the Schur–Szegö composition of polynomials, C. R. Acad. Sci. Paris, Ser. I 343 (2006) 81–86.