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Numerical Analysis

A new approach for approximating linear elasticity problems

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Dedicated to Professor Robert Dautray on the occasion of his 80th birthday

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Abstract

In this Note, we present and analyze a new method for approximating linear elasticity problems in dimension two or three. This approach directly provides approximate strains, i.e., without simultaneously approximating the displacements, in finite element spaces where the Saint Venant compatibility conditions are exactly satisfied in a weak form. ***To cite this article: P.G. Ciarlet, P. Ciarlet, Jr., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Une nouvelle approche pour approcher les problèmes d'élasticité linéaire. Dans cette Note, nous présentons et analysons une nouvelle méthode d'approximation de problèmes d'élasticité linéaire en dimension deux ou trois. Cette approche fournit directement des approximations des déformations, c'est-à-dire sans approcher simultanément les déplacements, dans des espaces d'éléments finis où les conditions de compatibilité de Saint Venant sont exactement satisfaites dans un sens faible. ***Pour citer cet article : P.G. Ciarlet, P. Ciarlet, Jr., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

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1. Intrinsic linearized elasticity

Greek indices range over the set $\{1, 2\}$, Latin indices range over the set $\{1, 2, 3\}$, and, unless otherwise specified, the summation convention with respect to repeated indices is used in conjunction with these rules. The Euclidean inner product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and norm of $\mathbf{a} \in \mathbb{R}^3$ are denoted $\mathbf{a} \cdot \mathbf{b}$ and $|\mathbf{a}|$. For $N \geqslant 2$, the matrix inner product of two $N \times N$ matrices $\boldsymbol{\varepsilon}$ and $\boldsymbol{e}$ is denoted $\boldsymbol{\varepsilon} : \boldsymbol{e}$, and $\mathbb{S}^N$ denotes the set of all $N \times N$ symmetric matrices. The identity mapping of a set X is denoted $\mathbf{id}_X$. The restriction of a mapping f to a set X is denoted $f|_X$. If V is a vector space and R is a subspace of V, the quotient space of V modulo R is denoted V/R and the equivalence class of $v \in V$ modulo R is denoted $\dot{v}$. The duality pairing between a topological vector space X and its dual X' is denoted ${}_{X'}\langle \cdot , \cdot \rangle_X$.

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Let x_i denote the coordinates of a point $x \in \mathbb{R}^3$, let $\partial_i := \partial/\partial x_i$ and $\partial_{ij} := \partial^2/\partial x_i \partial x_j$. Given a smooth enough vector field $\mathbf{v} = (v_i)$, we let $\nabla\mathbf{v} := (\partial_j v_i)$. Similar definitions hold in $\mathbb{R}^2$. Vector fields and spaces of vector fields are denoted by boldface letters. We let $\mathbb{L}^2_s(\Omega) = L^2(\Omega; \mathbb{S}^N)$ for $N = 2, 3$.

A *domain* in $\mathbb{R}^N$, $N \geqslant 2$, is an open, bounded, and connected subset of $\mathbb{R}^N$ whose boundary is Lipschitz-continuous. Let Ω be a domain in $\mathbb{R}^3$. Given any vector field $\mathbf{v} \in \mathbf{H}^1(\Omega)$, viewed here as a *displacement field* of the set Ω, let

$$\nabla_s \mathbf{v} := \frac{1}{2}(\nabla\mathbf{v}^T + \nabla\mathbf{v}) \in \mathbb{L}^2_s(\Omega)$$

denote its associated *linearized strain tensor field*. Let

$$\mathbf{R}(\Omega) := \{\mathbf{r} \in \mathbf{H}^1(\Omega);\ \nabla_s\mathbf{r} = \mathbf{0} \text{ in } \Omega\} = \{\mathbf{r} = \mathbf{c} + \mathbf{d} \wedge \mathbf{id}_\Omega;\ \mathbf{c} \in \mathbb{R}^3, \mathbf{d} \in \mathbb{R}^3\} \tag{1}$$

denote the space of *infinitesimal rigid displacement fields* of the set Ω.

Consider a *linearly elastic body* with $\overline{\Omega}$ as its reference configuration in the absence of applied forces. The elastic material constituting the body is characterized by its elasticity tensor $\mathbf{A} = (A_{ijkl})$, whose elements $A_{ijkl} = A_{jikl} = A_{klij} \in L^\infty(\Omega)$ are such that there exists a constant $\alpha > 0$ such that

$$\alpha\mathbf{t} : \mathbf{t} \leqslant \mathbf{A}(x)\mathbf{t} : \mathbf{t} \quad \text{for almost all } x \in \Omega \text{ and all } \mathbf{t} \in \mathbb{S}^3, \tag{2}$$

where $(\mathbf{A}(x)\mathbf{t})_{ij} := A_{ijkl}(x)t_{kl}$. Then the associated *pure traction problem* classically consists in finding a *displacement field* $\dot{\mathbf{u}} \in \dot{\mathbf{H}}^1(\Omega) := \mathbf{H}^1(\Omega)/\mathbf{R}(\Omega)$ that satisfies

$$J(\dot{\mathbf{u}}) = \inf_{\dot{\mathbf{v}} \in \dot{\mathbf{H}}^1(\Omega)} J(\dot{\mathbf{v}}) \quad \text{with } J(\dot{\mathbf{v}}) := \frac{1}{2}\int \mathbf{A}\nabla_s\mathbf{v} : \nabla_s\mathbf{v}\,\mathrm{d}x - L(\dot{\mathbf{v}}), \tag{3}$$

where $L : \mathbf{H}^1(\Omega) \to \mathbb{R}$ is a continuous linear form that takes into account the *applied forces* and satisfies the *compatibility condition* $L(\mathbf{r}) = 0$ for all $\mathbf{r} \in \mathbf{R}(\Omega)$. It is well known that the minimization problem (3) has one and only one solution, thanks to *Korn's inequality* [7].

By contrast, the *intrinsic approach* to the same problem consists in directly seeking the *linearized strain tensor field* $\boldsymbol{\varepsilon} := \nabla_s\dot{\mathbf{u}} \in \mathbb{L}^2_s(\Omega)$, which thus becomes the *primary unknown*, instead of the displacement field in the classical approach. The mathematical justification of such an approach crucially hinges on the following theorems, due to [4]. For any matrix field $\mathbf{e} = (e_{ij}) \in \mathcal{D}'(\Omega; \mathbb{S}^3)$, the matrix field $\mathbf{CURL\,CURL\,e} \in \mathcal{D}'(\Omega; \mathbb{S}^3)$ is defined by $(\mathbf{CURL\,CURL\,e})_{ij} = \boldsymbol{\varepsilon}_{ikl}\boldsymbol{\varepsilon}_{jmn}\partial_{ln}e_{km}$, where $(\boldsymbol{\varepsilon}_{ijk})$ denotes the orientation tensor.

Theorem 1. *Let Ω be a simply-connected domain in $\mathbb{R}^3$ and let $\mathbf{e} \in \mathbb{L}^2_s(\Omega)$ be a tensor field that satisfies*

$$\mathbf{CURL\,CURL\,e} = \mathbf{0} \quad \text{in } H^{-2}(\Omega; \mathbb{S}^3). \tag{4}$$

Then there exists a vector field $\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\nabla_s\mathbf{v} = \mathbf{e}$ in $\mathbb{L}^2_s(\Omega)$, and all the other solutions $\tilde{\mathbf{v}}$ to the equation $\nabla_s\tilde{\mathbf{v}} = \mathbf{e}$ are of the form $\tilde{\mathbf{v}} = \mathbf{v} + \mathbf{r}$ for some $\mathbf{r} \in \mathbf{R}(\Omega)$.

As shown in [1], the six scalar equations contained in the matrix equation (4) are equivalent to the following equations, which constitute *a weak version of the classical Saint Venant compatibility conditions* on the components e_{ij} of the tensor field $\mathbf{e}$:

$$\partial_{lj}e_{ik} + \partial_{ki}e_{jl} - \partial_{li}e_{jk} - \partial_{kj}e_{il} = 0 \quad \text{in } H^{-2}(\Omega) \text{ for all } i, j, k, l.$$

Theorem 2. *Let Ω be a simply-connected domain in $\mathbb{R}^3$. Define the space*

$$\mathbb{E}(\Omega) := \{\mathbf{e} \in \mathbb{L}^2_s(\Omega);\ \mathbf{CURL\,CURL\,e} = \mathbf{0} \text{ in } H^{-2}(\Omega; \mathbb{S}^3)\}, \tag{5}$$

and, given any $\mathbf{e} \in \mathbb{E}_s(\Omega)$, let $\dot{\mathbf{v}} = \mathcal{F}(\mathbf{e})$ denote the unique element in the space $\dot{\mathbf{H}}^1(\Omega)$ that satisfies $\mathbf{e} = \nabla_s\dot{\mathbf{v}}$ (Theorem 1). *Then the linear mapping $\mathcal{F} : \mathbb{E}(\Omega) \to \dot{\mathbf{H}}^1(\Omega)$ defined in this fashion is an isomorphism between the Hilbert spaces $\mathbb{E}(\Omega)$ and $\dot{\mathbf{H}}^1(\Omega)$.*

Thanks to the isomorphism $\mathcal{F}$, the minimization problem (3) can be recast as another minimization problem (see (6) below), where $\mathbf{e}(\dot{\mathbf{u}})$ is now the primary unknown.

Theorem 3. *Let Ω be a simply-connected domain in $\mathbb{R}^3$. Then the minimization problem: Find $\boldsymbol{\varepsilon} \in \mathbb{E}(\Omega)$ such that*

$$j(\boldsymbol{\varepsilon}) = \inf_{\mathbf{e} \in \mathbb{E}(\Omega)} j(\mathbf{e}), \quad \text{where } j(\mathbf{e}) := \frac{1}{2}\int_\Omega \mathbf{Ae} : \mathbf{e}\, \mathrm{d}x - (L \circ \mathcal{F})(\mathbf{e}), \tag{6}$$

has one and only solution $\boldsymbol{\varepsilon}$. Besides $\boldsymbol{\varepsilon} = \nabla_s \dot{\mathbf{u}}$, where $\dot{\mathbf{u}}$ is the unique solution to the minimization problem (3).

2. A curl curl-free finite element space

The objective of this Note is to describe and analyze a direct finite element approximation of the minimization problem (6). One noticeable feature of this approach is that it overcomes the difficulties traditionally associated with the desired *symmetry* of the approximated tensor field; in this direction, see the illuminating discussions in [2] and [3].

For simplicity, we restrict ourselves here to the two-dimensional case, in which case the relation $\mathbf{CURL\,CURL\,e} = \mathbf{0}$ in $H^{-2}(\Omega; \mathbb{S}^3)$, which appears in (4) and (5), is to be replaced by the single equation

$$\operatorname{curl} \mathbf{curl}\, \mathbf{e} = \partial_{11} e_{22} - 2\partial_{12} e_{12} + \partial_{22} e_{11} = 0 \quad \text{in } H^{-2}(\Omega). \tag{7}$$

Complete proofs and further extensions, notably to dimension 3, will be found in [5] and [6].

In what follows, $P_k(T; \mathbb{X})$ denotes the space of all mappings from a subset T of $\mathbb{R}^N$ into a vector space $\mathbb{X}$, whose components are polynomials of degree $\leqslant k$ in the variables x_α if $N = 2$, or x_i if $N = 3$.

To begin with, we describe a finite element, which provides a new type of *edge finite element*, in the sense of Nédélec [8]. The length element is denoted dl.

Theorem 4. *Let T be a non-degenerate triangle with edges s_i, $1 \leqslant i \leqslant 3$. Given any edge s_i of T, let τ_i denote a unit vector parallel to s_i, and let the degrees of freedom d_i, $1 \leqslant i \leqslant 3$, be defined as*

$$d_i(\mathbf{e}) := \int_{s_i} \boldsymbol{\tau}_i \cdot \mathbf{e} \boldsymbol{\tau}_i \, \mathrm{d}l \quad (\textit{no summation w.r.t. } i) \textit{ for all } \mathbf{e} \in P_0(T; \mathbb{S}^2). \tag{8}$$

Then the set $\{d_i; 1 \leqslant i \leqslant 3\}$ is $P_0(T; \mathbb{S}^2)$-unisolvent, i.e., a tensor field $\mathbf{e} \in P_0(T; \mathbb{S}^2)$ is uniquely defined by the three numbers $d_i(\mathbf{e})$, $1 \leqslant i \leqslant 3$.

Sketch of proof. The space $P_0(T; \mathbb{S}^2)$ is of dimension three and there are three degrees of freedom d_i. It thus suffices to show that, if $\mathbf{e} = (e_{\alpha\beta}) \in P_0(T; \mathbb{S}^2)$ satisfies $d_i(\mathbf{e}) = 0$, $1 \leqslant i \leqslant 3$, then $\mathbf{e} = \mathbf{0}$ on T. Since the functions $e_{\alpha\beta}$ are constant on T, and the length of each edge s_i is > 0, the relations $d_i(\mathbf{e}) = 0$ are equivalent to the linear system $\tau^i_\alpha \tau^i_\beta e_{\alpha\beta} = 0$ (no summation w.r.t. i), $1 \leqslant i \leqslant 3$, where τ^i_α designates the α-th coordinate of the vector $\boldsymbol{\tau}^i$. One can then show that the 3×3 matrix of this linear system is invertible if and only if T is a non-degenerate triangle. Therefore $(e_{\alpha\beta}) = \mathbf{e} = \mathbf{0}$. □

If T is a non-degenerate *tetrahedron* in $\mathbb{R}^3$, one can similarly show that the set $\{d_i; 1 \leqslant i \leqslant 6\}$, where d_i is again defined as in (8) along each edge s_i, $1 \leqslant i \leqslant 6$, of T, is $P_0(T; \mathbb{S}^3)$-unisolvent; cf. [5].

From now on, Ω denotes a *polygonal domain* in $\mathbb{R}^2$, and we consider *triangulations* $\mathcal{T}^h$ of the set $\overline{\Omega}$ by triangles $T \in \mathcal{T}^h$ subjected to the usual conditions; in particular, all the triangles $T \in \mathcal{T}^h$ are non-degenerate. Given such a triangulation $\mathcal{T}^h$, let Σ^h denote the set of all *'interior' edges* found in $\mathcal{T}^h$ (i.e., that are not contained in the boundary $\partial\Omega$), let Σ^h_∂ denote the set of all '*boundary' edges* found in $\mathcal{T}^h$ (i.e., that are contained in $\partial\Omega$), and let A^h denote the set of all *'interior' vertices* found in $\mathcal{T}^h$ (i.e., that are contained in Ω). We also assume that each interior or boundary edge $\sigma \in \Sigma^h \cup \Sigma^h_\partial$ is *oriented.*

Theorem 5. *Given any triangulation $\mathcal{T}^h$ of $\overline{\Omega}$, define the finite element space*

$$\widetilde{\mathbb{E}}^h := \Big\{\mathbf{e}^h \in \mathbb{L}^2_s(\Omega); \mathbf{e}^h|_T \in P_0(T; \mathbb{S}^2) \textit{ for all } T \in \mathcal{T}^h \textit{ and}$$
$$\int_\sigma \boldsymbol{\tau} \cdot (\mathbf{e}^h|_{T_1})\boldsymbol{\tau}\, \mathrm{d}l = \int_\sigma \boldsymbol{\tau} \cdot (\mathbf{e}^h|_{T_2})\boldsymbol{\tau}\, \mathrm{d}l \textit{ for all } \sigma = T_1 \cap T_2 \in \Sigma^h \textit{ with } T_1, T_2 \in \mathcal{T}^h\Big\}. \tag{9}$$

Then each tensor field $\mathbf{e}^h \in \widetilde{\mathbb{E}}^h$ *is uniquely defined by the numbers* $d_\sigma(\mathbf{e}_h)$, $\sigma \in \Sigma^h \cup \Sigma^h_\partial$, *where the degrees of freedom* $d_\sigma : \widetilde{\mathbb{E}}^h \to \mathbb{R}$ *are defined by*

$$d_\sigma(\mathbf{e}_h) := \begin{cases} \int_\sigma \boldsymbol{\tau} \cdot (\mathbf{e}^h|_{T_1})\boldsymbol{\tau}\,\mathrm{d}l = \int_\sigma \boldsymbol{\tau} \cdot (\mathbf{e}^h|_{T_2})\boldsymbol{\tau}\,\mathrm{d}l & \text{if } \sigma = T_1 \cap T_2 \in \Sigma^h, \\ \int_\sigma \boldsymbol{\tau} \cdot (\mathbf{e}^h|_{\sigma})\boldsymbol{\tau}\,\mathrm{d}l & \text{if } \sigma \in \Sigma^h_\partial. \end{cases} \tag{10}$$

Furthermore, given any interior vertex $a \in A^h$, *let* $\{T; T \in \mathcal{T}^h(a)\}$ *denote the set formed by all the triangles of* $\mathcal{T}^h$ *that have the vertex a in common, and let* $\widetilde{\Omega} = \widetilde{\Omega}(a, \sigma) := \mathrm{int}(\bigcup_{T \in \mathcal{T}^h(a)} T - \sigma)$, *where* σ *is any one of the interior edges that have a as an end-point. Then*

$$\operatorname{curl}\mathbf{curl}\,\mathbf{e}^h = 0 \quad \text{in } \mathcal{D}'(\widetilde{\Omega}) \text{ for all } \mathbf{e}^h \in \widetilde{\mathbb{E}}^h. \tag{11}$$

Sketch of proof. That each tensor field $\mathbf{e}^h \in \widetilde{\mathbb{E}}^h$ is uniquely defined by the numbers $d_\sigma(\mathbf{e}^h)$, $\sigma \in \Sigma^h \cup \Sigma^h_\partial$ follows from the unisolvence established in Theorem 4.

Let T_1 and T_2 be two adjacent triangles with a common edge $\sigma = T_1 \cap T_2 = [a, b] \in \Sigma^h$, let $\boldsymbol{\nu} = (\nu_\alpha)$ denote the unit outer normal vector to T_1 along σ, and assume (to fix ideas) that σ is oriented with $\boldsymbol{\tau} = (\tau_\alpha)$ with $\tau_1 = -\nu_2$ and $\tau_2 = \nu_1$. For any function $\varphi \in \mathcal{D}(\Omega)$ such that $\operatorname{supp}\varphi \subset \widehat{\Omega} := \mathrm{int}(T_1 \cup T_2)$,

$$_{D'(\widehat{\Omega})}\langle \operatorname{curl}\mathbf{curl}\,\mathbf{e}^h, \varphi\rangle_{D(\widehat{\Omega})} = \int_\sigma \big\{\big([e_{22}]\tau_2 + [e_{12}]\tau_1\big)\partial_1\varphi - \big([e_{21}]\tau_2 + [e_{11}]\tau_1\big)\partial_2\varphi\big\}\,\mathrm{d}l,$$

where $[e_{\alpha\beta}] := (e_{\alpha\beta}|_{T_1})|_\sigma - (e_{\alpha\beta}|_{T_2})|_\sigma$. This relation, combined with the relation $[e_{\alpha\beta}]\tau_\alpha\tau_\beta = 0$ along σ (which follows from the definition of the spaces $\widetilde{\mathbb{E}}^h$; cf. (9)) then gives

$$_{D'(\widehat{\Omega})}\langle \operatorname{curl}\mathbf{curl}\,\mathbf{e}^h, \varphi\rangle_{D(\widehat{\Omega})} = \frac{1}{\tau_1}\big([e_{12}]\tau_1 + [e_{22}]\tau_2\big)\int_\sigma (\tau_1\partial_1\varphi + \tau_2\partial_2\varphi)\,\mathrm{d}l \quad \text{if } \tau_1 \neq 0,$$

$$_{D'(\widehat{\Omega})}\langle \operatorname{curl}\mathbf{curl}\,\mathbf{e}^h, \varphi\rangle_{D(\widehat{\Omega})} = -(\operatorname{sign}\tau_2)[e_{21}]\int_\sigma \partial_2\varphi\,\mathrm{d}l \quad \text{if } \tau_1 = 0.$$

Since both integrals along σ vanish (their absolute values are equal to $|\varphi(b) - \varphi(a)|$), it follows that $_{D'(\widehat{\Omega})}\langle \operatorname{curl}\mathbf{curl}\,\mathbf{e}^h, \varphi\rangle_{D(\widehat{\Omega})} = 0$. Hence $\operatorname{curl}\mathbf{curl}\,\mathbf{e}^h = 0$ in $\mathcal{D}'(\widehat{\Omega})$. The set $\widetilde{\Omega} = \widetilde{\Omega}(a, \sigma)$ being defined as in the statement of Theorem 5, a similar argument shows that $_{\mathcal{D}'(\widetilde{\Omega})}\langle \operatorname{curl}\mathbf{curl}\,\mathbf{e}^h, \varphi\rangle_{\mathcal{D}(\widetilde{\Omega})} = 0$ for all $\varphi \in \mathcal{D}(\widetilde{\Omega})$. Therefore, $\operatorname{curl}\mathbf{curl}\,\mathbf{e}^h = 0$ in $\mathcal{D}'(\widetilde{\Omega})$. □

The definition (9) of the finite element space $\widetilde{\mathbb{E}}^h$ and the definition (10) of its degrees of freedom d_σ together imply that the *dimension of* $\widetilde{\mathbb{E}}^h$ is equal to the number of edges in $\mathcal{T}^h$ and that the *support of the associated basis functions in* $\widetilde{\mathbb{E}}^h$ is either the union of the two adjacent triangles having the edge σ in common if $\sigma \in \Sigma^h$, or a single triangle if $\sigma \in \Sigma^h_\partial$.

The next theorem shows how to transform the finite element space $\widetilde{\mathbb{E}}^h$ into a curl **curl**-free one, denoted $\mathbb{E}^h$ (cf. (12)), by adding an ad hoc *constraint* $\varphi_a(\mathbf{e}^h) = 0$ at each interior vertex $a \in A^h$. Note that the *explicit form* of the linear forms φ_a can be easily computed; cf. [5].

Theorem 6. *Given any interior vertex* $a \in A^h$, *there exists a linear combination* $\varphi_a : \widetilde{\mathbb{E}}^h \to \mathbb{R}$ *of the degrees of freedom along the edges of the triangles* $T \in \mathcal{T}^h$ *having a as a common vertex, with the following property*:

$$\operatorname{curl}\mathbf{curl}\,\mathbf{e}^h = 0 \quad \text{in } \mathcal{D}'(\Omega) \text{ for all } \mathbf{e}^h \in \mathbb{E}^h := \big\{\mathbf{e}^h \in \widetilde{\mathbb{E}}^h;\ \varphi_a(\mathbf{e}^h) = 0 \text{ for all } a \in A^h\big\}. \tag{12}$$

If, in addition, the polygonal domain Ω *is simply-connected, the space* $\mathbb{E}^h$ *can be equivalently defined as*

$$\mathbb{E}^h = \big\{\boldsymbol{\nabla}_s\dot{\mathbf{v}}^h \in \mathbb{L}^2_s(\Omega);\ \dot{\mathbf{v}}^h \in \dot{\mathbf{V}}^h\big\}, \tag{13}$$

where, the space $\mathbf{R}(\Omega)$ *being now defined as the two-dimensional analog of* (1),

$$\dot{\mathbf{V}}^h = \mathbf{V}^h/\mathbf{R}(\Omega) \quad \text{with } \mathbf{V}^h := \big\{\mathbf{v}^h \in \mathbf{C}^0(\overline{\Omega});\ \mathbf{v}^h|_T \in P_1(T; \mathbb{R}^2)\big\}. \tag{14}$$

Sketch of proof. (i) Consider first the case where there is *only one interior vertex* a in the triangulation. In this case, where $\dim \widetilde{\mathbb{E}}^h = \dim \mathbb{E}^h + 1$, relation (11) and the two-dimensional version of Theorem 1 together implies that there exists a linear form φ_a of the announced form such that $\{\mathbf{e}^h \in \widetilde{\mathbb{E}}^h; \varphi_a(\mathbf{e}^h) = 0\} = \{\boldsymbol{\nabla}_s \dot{\mathbf{v}}^h \in \mathbb{L}^2_s(\Omega); \dot{\mathbf{v}}^h \in \dot{\mathbf{V}}^h\}$. Hence $\operatorname{curl} \mathbf{curl}\, \mathbf{e}^h = 0$ in $\mathcal{D}'(\Omega)$ for all $\mathbf{e}^h \in \mathbb{E}^h$ in this case, since $\operatorname{curl} \mathbf{curl}\, \boldsymbol{\nabla}_s \mathbf{v} = 0$ in $\mathcal{D}'(\Omega)$ for any vector field $\mathbf{v} \in \mathcal{D}'(\Omega; \mathbb{R}^2)$.

(ii) In the general case, the conditions $\varphi_a(\mathbf{e}^h) = 0$ satisfied by each $\mathbf{e}^h \in \mathbb{E}^h$ at all $a \in A^h$ imply that, given any point $x \in \Omega$, the distribution $\operatorname{curl} \mathbf{curl}\, \mathbf{e}^h$ vanishes in an open set containing x by (i). Hence $\operatorname{curl} \mathbf{curl}\, \mathbf{e}^h = 0$ in $\mathcal{D}'(\Omega)$ by the "principle of localization of distributions"; cf. Schwartz [9, Chapter 1, Section 3].

(iii) Assume that Ω is simply-connected, and let $\mathbf{e}^h = (e^h_{\alpha\beta}) \in \mathbb{E}^h$ be given. Since $\mathbf{e}_h \in \mathbb{L}^2_s(\Omega)$ satisfies $\operatorname{curl} \mathbf{curl}\, \mathbf{e}^h = 0$ in $H^{-2}(\Omega)$ by (ii), the two-dimensional analog of Theorem 1 shows that there exists a vector field $\dot{\mathbf{v}}^h = (\dot{v}^h_\alpha) \in \mathbf{H}^1(\Omega)/\mathbf{R}(\Omega)$ such that $\mathbf{e}^h = \boldsymbol{\nabla}_s \dot{\mathbf{v}}^h$ in $\mathbb{L}^2_s(\Omega)$. Since, for each $T \in \mathcal{T}^h$,

$$\partial_{\alpha\beta} v^h_\tau = \partial_\alpha e_{\tau\beta}(\mathbf{v}^h) + \partial_\beta e_{\tau\alpha}(\mathbf{v}^h) - \partial_\tau e_{\alpha\beta}(\mathbf{v}^h) \quad \text{in } H^{-1}(\operatorname{int} T),$$

and $\mathbf{e}^h|_T \in P_0(T; \mathbb{S}^2)$, it follows that $\mathbf{v}^h|_T \in P_1(T; \mathbb{R}^2)$. Consequently, $\mathbf{v}^h \in \mathbf{C}^0(\overline{\Omega})$ since $\mathbf{v}^h \in \mathbf{H}^1(\Omega)$. We have thus shown that

$$\mathbb{E}^h \subset \widehat{\mathbb{E}}^h := \{\boldsymbol{\nabla}_s \dot{\mathbf{v}}^h \in \mathbb{L}^2_s(\Omega); \dot{\mathbf{v}}^h \in \dot{\mathbf{V}}^h\}. \tag{15}$$

Noting that, for a simply-connected polygonal domain, $\dim \widetilde{\mathbb{E}}^h =$ [number of edges], and that $\dim \widehat{\mathbb{E}}^h = [2 \times$ (number of vertices) $- 3]$, and using Euler's relation, we infer that $\dim \widetilde{\mathbb{E}}^h - \dim \widehat{\mathbb{E}}^h =$ [number of interior vertices]. Consequently,

$$\dim \mathbb{E}^h = \dim \widehat{\mathbb{E}}^h. \tag{16}$$

Relation (13) then follows from relations (15) and (16). □

Theorem 6 shows that there exists an isomorphism between the spaces $\mathbb{E}^h$ and $\dot{\mathbf{V}}^h$, which is nothing but the discrete analog of the isomorphism $\mathcal{F}$ between the spaces $\mathbb{E}(\Omega)$ and $\mathbf{H}^1(\Omega)$ (Theorem 2).

3. The discrete problem; convergence

In what follows, Ω is again assumed to be a simply-connected polygonal domain in $\mathbb{R}^2$. The *discrete problem* is now defined, as the minimization problem (17).

Theorem 7. *Given any triangulation $\mathcal{T}_h$ of $\overline{\Omega}$, let $\mathbb{E}^h$ be the finite element space defined in* (12). *Then there exists one and only one $\boldsymbol{\varepsilon}^h \in \mathbb{E}^h$ such that*

$$j(\boldsymbol{\varepsilon}^h) = \inf_{\mathbf{e}^h \in \mathbb{E}^h} j(\mathbf{e}^h), \tag{17}$$

where j is the functional defined in (6). *Let $\dot{\mathbf{V}}^h$ be the finite element space defined in* (14), *and let J be the functional defined in* (3). *Then $\boldsymbol{\varepsilon}^h = \boldsymbol{\nabla}_s \dot{\mathbf{u}}^h$, where $\dot{\mathbf{u}}^h \in \dot{\mathbf{V}}^h$ is the unique solution to the minimization problem $J(\dot{\mathbf{u}}^h) = \inf_{\dot{\mathbf{v}}^h \in \dot{\mathbf{V}}^h} J(\dot{\mathbf{v}}^h)$.*

Proof. We have $j(\mathbf{e}) = \frac{1}{2} b(\mathbf{e}, \mathbf{e}) - l(\mathbf{e})$ for all $\mathbf{e} \in \mathbb{L}^2_s(\Omega)$, where the bilinear form b and the linear form l satisfy all the assumptions of the Lax–Milgram lemma over the space $\mathbb{E}(\Omega)$ of (5) (thanks in particular to the inequality (2)), hence over its subspace $\mathbb{E}^h$, which is closed ($\mathbb{E}^h$ is finite-dimensional). Consequently, there exists one, and only one, minimizer $\boldsymbol{\varepsilon}^h$ of the functional j over $\mathbb{E}^h$.

That $\dot{\mathbf{u}}^h$ minimizes the functional J over $\dot{\mathbf{V}}^h$ implies that $\boldsymbol{\nabla}_s \dot{\mathbf{u}}^h$ minimizes the functional j over $\mathbb{E}^h$ since $\boldsymbol{\nabla}_s \dot{\mathbf{u}}^h \in \mathbb{E}^h$ by Theorem 6. Hence $\boldsymbol{\varepsilon}^h = \boldsymbol{\nabla}_s \dot{\mathbf{u}}^h$ since the minimizer is unique. □

Finally, we examine the *convergence* of the method.

Theorem 8. *Consider a regular family of triangulations $\mathcal{T}^h$ of $\overline{\Omega}$. Then*

$$\|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^h\|_{\mathbb{L}^2_s(\Omega)} \to 0 \quad \textit{as } h \to 0.$$

If $\mathbf{u} \in \mathbf{H}^2(\Omega)$*, there exists a constant* C *independent of* h *such that*

$$\|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^h\|_{\mathbb{L}^2_s(\Omega)} \leqslant C\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} h.$$

Proof. Let b denote the bilinear form that appears in the functional j. Observing that $\boldsymbol{\varepsilon}^h$ is the projection of $\boldsymbol{\varepsilon}$ onto $\mathbb{E}^h$ with respect to the inner product b and taking into account the assumptions made in Section 1 on the elasticity tensor A, we infer that there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned} C_1\|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^h\|^2_{\mathbb{L}^2_s(\Omega)} &\leqslant b(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^h, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^h) = \inf_{\mathbf{e}^h \in \mathbb{E}^h} b(\boldsymbol{\varepsilon} - \mathbf{e}^h, \boldsymbol{\varepsilon} - \mathbf{e}^h) \\ &\leqslant C_2 \inf_{\mathbf{e}^h \in \mathbb{E}^h} \|\boldsymbol{\varepsilon} - \mathbf{e}^h\|^2_{\mathbb{L}^2_s(\Omega)} = C_2 \inf_{\mathbf{v}^h \in \mathbf{V}^h} \|\boldsymbol{\nabla}_s\mathbf{u} - \boldsymbol{\nabla}_s\mathbf{v}^h\|^2_{\mathbb{L}^2_s(\Omega)}. \end{aligned}$$

The conclusions then follow by standard error estimates. □

Remark. A *non-conforming method* would consist in using the space $\widetilde{\mathbb{E}}^h$ of (9) instead of the space $\mathbb{E}^h$.

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