



Algebraic Geometry

A note on congruences for theta divisors

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Abstract

The classes of two theta divisors on an Abelian variety in the naive Grothendieck ring of varieties need not be congruent modulo the class of the affine line. **To cite this article:** *F. Heinloth, C. R. Acad. Sci. Paris, Ser. I 346 (2008)*.

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Résumé

Une Note sur les congruences des diviseurs thêta. Dans l'anneau de Grothendieck des variétés, les classes de deux diviseurs thêta d'une même variété abélienne ne sont pas nécessairement congruentes modulo la classe de la droite affine. **Pour citer cet article :** *F. Heinloth, C. R. Acad. Sci. Paris, Ser. I 346 (2008)*.

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Soient Θ, Θ' deux diviseurs thêta sur une variété abélienne définie sur un corps fini \mathbb{F}_q . Berthelot, Bloch et Esnault démontrent dans [2] que $|\Theta(\mathbb{F}_q)| \equiv |\Theta'(\mathbb{F}_q)| \pmod{q}$.

Plus généralement, Serre a conjecturé que sur une variété abélienne définie sur un corps k , « les motifs de deux diviseurs thêta diffèrent par un multiple du motif de Lefschetz ».

Dans cette Note, nous posons une question plus élémentaire :

Question. Les diviseurs $[\Theta]$ et $[\Theta']$ sont-ils congruents modulo la classe \mathbb{L} de la droite affine dans l'anneau de Grothendieck $K_0(\text{Var}_k)/(\mathbb{L})$ des variétés sur k ?

L'existence de produits de courbes elliptiques qui sont des Jacobiennes fournit une surface abélienne $E \times E'$ qui contient deux diviseurs thêta, l'un Θ provenant de la courbe, et l'autre Θ' étant défini par $E \times \Theta' + 0 \times E'$. En caractéristique zéro, $K_0(\text{Var}_k)/(\mathbb{L})$ est isomorphe au groupe libre engendré par les classes de variétés projectives, lisses et irréductibles qui sont stablement birationnelles. On en déduit que les classes des deux diviseurs thêta ne sont pas les mêmes dans $K_0(\text{Var}_k)/(\mathbb{L})$. Ceci montre que la réponse à la question est « non » en général.

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En caractéristique zéro considérons l’homomorphisme $\chi_{\text{mot}} : K_0(\text{Var}_k) \rightarrow K_0(\text{CM}_k)$ à valeurs dans l’anneau de Grothendieck des motifs effectifs de Chow sur k , qui envoie la classe d’une variété projective et lisse sur la classe $[h(X)]$ de son motif de Chow. Alors $\chi_{\text{mot}}([\Theta]) - \chi_{\text{mot}}([\Theta']) = -[\mathbb{L}_{\text{mot}}]$, où \mathbb{L}_{mot} est le motif de Lefschetz. On voit que cet exemple simple vérifie la conjecture de Serre. De façon plus précise, Serre s’attend à ce que la divisibilité soit vérifiée dans le K_0 des motifs construits à l’aide de l’équivalence algébrique.

1. Introduction

A *theta divisor* on an Abelian variety A over a field is an effective ample divisor Θ such that the global sections of the line bundle $\mathcal{O}_A(\Theta)$ are one-dimensional. For example, the image of the $(g - 1)$ st symmetric power of a smooth projective curve C of genus g with a rational point in its Jacobian is a theta divisor.

Berthelot, Bloch and Esnault prove the following [2, Theorem 1.4]:

Theorem 1.1. *Let Θ, Θ' be two theta divisors on an Abelian variety, all defined over a finite field \mathbb{F}_q . Then*

$$|\Theta(\mathbb{F}_q)| \equiv |\Theta'(\mathbb{F}_q)| \pmod{q}.$$

As the Lefschetz motive over a finite field \mathbb{F}_q has q points, the theorem by Berthelot, Bloch and Esnault is the point counting consequence of a more general prediction by Serre saying that on an Abelian variety defined over a field k , “the motives of two theta divisors Θ and Θ' differ by a multiple of the Lefschetz motive”. In this Note, we ask a more elementary question:

Question 1.2. Is $[\Theta] \equiv [\Theta'] \pmod{\mathbb{L}}$ in $K_0(\text{Var}_k)$?

Here $K_0(\text{Var}_k)$ denotes the naive Grothendieck ring of varieties over k , i.e. the free Abelian group on isomorphism classes $[X]$ of varieties over k modulo the relations $[X] = [X - Y] + [Y]$ for $Y \subset X$ a closed subvariety. The ring structure is given by the product of varieties, and \mathbb{L} is the class $[\mathbb{A}_k^1]$ of the affine line.

We give a negative answer to the above question using the fact that products of two elliptic curves can be Jacobians.

2. A counterexample on a product of two elliptic curves

In [5], Hayashida and Nishi show that certain products of two elliptic curves $E \times E'$ are Jacobians. More precisely they prove (Theorem 4 and Proposition 4):

Theorem 2.1. *Let E and E' be isogenous elliptic curves with endomorphism ring \mathcal{O} the ring of integers in the number field $\mathbb{Q}(\sqrt{-m})$, where m is a nonnegative integer. Then $E \times E'$ is a Jacobian variety of a curve of genus 2 if E' is not isomorphic to E or if $m \notin \{0, 1, 3, 7, 15\}$.*

Recall that two smooth projective varieties X and X' are called *stably birational*, if $X \times \mathbb{P}^n$ and $X' \times \mathbb{P}^{n'}$ are birational for some integers n and n' . Denote the set of stably birational classes of irreducible smooth projective varieties over k by SB_k , and denote the free Abelian group on SB_k by $\mathbb{Z}[\text{SB}_k]$. It carries a commutative ring structure induced by the product of varieties.

The weak factorization theorem by Włodarczyk and Abramovich et al. (see [7] and [1]) implies that two birational irreducible smooth projective varieties over a field of characteristic zero are connected by a series of blow-ups and blow-downs along smooth centers. Larsen and Lunts in [6], using the weak factorization theorem, show that over a field k of characteristic zero the following holds:

Theorem 2.2. *There is a ring homomorphism $b : K_0(\text{Var}_k) \rightarrow \mathbb{Z}[\text{SB}_k]$ sending the class of an irreducible smooth projective variety $[X]$ to its stably birational class $[X]_b$.*

Obviously, this homomorphism sends the class \mathbb{L} of the affine line to zero. On the other hand, using the weak factorization theorem again, it is easy to see that two stably birational irreducible smooth projective varieties have the

same class in $\mathbf{K}_0(\text{Var}_k)/(\mathbb{L})$. Therefore, as pointed out in [6], b actually induces an isomorphism of $\mathbf{K}_0(\text{Var}_k)/(\mathbb{L})$ and $\mathbb{Z}[\text{SB}_k]$.

Now take Θ a genus two curve with a rational point over a field k of characteristic zero such that the Jacobian of Θ is $E \times E'$ as in Theorem 2.1. Let $\Theta' = E \times 0' + 0 \times E'$. Then $b([\Theta]) = [\Theta]_b \neq b([\Theta']) = [E]_b + [E']_b - [\text{Spec } k]_b$, as a curve of genus 2 is not stably birational to an elliptic curve (note that we are working in a *free* Abelian group).

Therefore the classes of Θ and Θ' are not congruent modulo \mathbb{L} in the Grothendieck ring of varieties.

Let us consider, when k has characteristic zero, the homomorphism $\chi_{\text{mot}} : \mathbf{K}_0(\text{Var}_k) \rightarrow \mathbf{K}_0(\text{CM}_k)$ constructed by Gillet and Soulé in [3] and by Guillen and Navarro Aznar in [4]. It sends the class of a smooth projective variety X to the class $[h(X)]$ of its Chow motive in the Grothendieck ring of effective Chow motives. The Chow motive of Θ is of the form $\mathbb{1} \oplus h^1(\Theta) \oplus \mathbb{L}_{\text{mot}}$, where \mathbb{L}_{mot} is the Lefschetz motive and $h^1(\Theta) \cong h^1(E \times E') \cong h^1(E) \oplus h^1(E')$, hence

$$\chi_{\text{mot}}([\Theta]) - \chi_{\text{mot}}([\Theta']) = [h(\Theta)] - [h(E)] - [h(E')] + 1 = -[\mathbb{L}_{\text{mot}}].$$

Hence in the simple example given above, the conjecture of Serre is verified. More precisely, Serre expects divisibility in the Grothendieck ring of motives with respect to algebraic equivalence.

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