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Partial Differential Equations/Complex Analysis

A flower structure of backward flow invariant domains for semigroups

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Abstract

In this Note, we study conditions which ensure the existence of backward flow invariant domains for semigroups of holomorphic self-mappings of a simply connected domain D. More precisely, the problem is the following. Given a one-parameter semigroup $\mathcal{S}$ on D, find a simply connected subset $\Omega \subset D$ such that each element of $\mathcal{S}$ is an automorphism of Ω, in other words, such that $\mathcal{S}$ forms a one-parameter group on Ω. ***To cite this article: M. Elin et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Une structure en rosace de domaines invariants par flot rétrograde de semi-groupes. Dans cette Note nous établissons des conditions qui assument l'existence de domaines invariants par flot rétrograde de semi-groupes d'applications holomorphes d'un domaine D, simplement connexe, dans lui-même. De manière plus précise, étant donné un semi-groupe $\mathcal{S}$ à un paramètre sur D, trouver un sous-ensemble connexe $\Omega \subset D$ tel que chaque élément de $\mathcal{S}$ soit un automorphisme de Ω, en d'autres termes tel que $\mathcal{S}$ soit un groupe à un paramètre sur Ω. ***Pour citer cet article : M. Elin et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Let D be a simply connected domain in the complex plane $\mathbb{C}$. By $\mathrm{Hol}(D, \Omega)$ we denote the set of all holomorphic functions on D with values in a domain Ω in $\mathbb{C}$. We write $\mathrm{Hol}(D)$ for $\mathrm{Hol}(D, D)$, the set of holomorphic self-mappings of D. This set is a topological semigroup with respect to composition. We denote by $\mathrm{Aut}(D)$ the group of all automorphisms of D; thus $F \in \mathrm{Aut}(D)$ if and only if F is univalent on D and $F(D) = D$.

Definition 1. A family $\mathcal{S} = \{F_t\}_{t \geqslant 0} \subset \mathrm{Hol}(D)$ is said to be a one-parameter continuous semigroup (semiflow) on D if:

(i) $F_t(F_s(z)) = F_{t+s}(z)$ for all $t, s \geqslant 0$;
(ii) $\lim_{t \to 0^+} F_t(z) = z$ for all $z \in D$.

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If, in addition, condition (i) holds for all $t, s \in \mathbb{R}$, then $(F_t)^{-1} = F_{-t}$ for each $t \in \mathbb{R}$; and $\mathcal{S}$ is called a *one-parameter continuous group* (*flow*) on D. In this case, $\mathcal{S} \subset \mathrm{Aut}(D)$.

In this Note, we study the following problem: **Given a one-parameter semigroup $\mathcal{S} \subset \mathrm{Hol}(D)$, find a simply connected domain $\Omega \subset D$ (if it exists) such that $\mathcal{S} \subset \mathrm{Aut}(\Omega)$.**

It follows by a result of E. Berkson and H. Porta [4] that each continuous semigroup is differentiable in $t \in \mathbb{R}^+ = [0, \infty)$, (see also [1] and [13]). So, for each continuous semigroup (semiflow) $\mathcal{S} = \{F_t\}_{t \geqslant 0} \subset \mathrm{Hol}(D)$, the limit,

$$\lim_{t \to 0^+} \frac{z - F_t(z)}{t} = f(z), \quad z \in D,$$

exists and defines a holomorphic mapping $f \in \mathrm{Hol}(D, \mathbb{C})$. This mapping f is called the **(infinitesimal) generator of** $\mathcal{S} = \{F_t\}_{t \geqslant 0}$.

Let now $D = \Delta$ be the open unit disk in $\mathbb{C}$.

Observe that if a semigroup $\mathcal{S} = \{F_t\}_{t \geqslant 0}$ does not contain an elliptic automorphism of Δ, then there is a unique point $\tau \in \bar{\Delta}$ which is the attractive point for the semigroup $\mathcal{S}$, i.e., for all $z \in \Delta$,

$$\lim_{t \to \infty} F_t(z) = \tau. \tag{1}$$

This point is usually referred as the **Denjoy–Wolff point** of $\mathcal{S}$. In addition,

- if $\tau \in \Delta$, then $\tau = F_t(\tau)$ is a unique fixed point of $\mathcal{S}$ in Δ;
- if $\tau \in \partial\Delta$, then $\tau = \lim_{r \to 1^-} F_t(r\tau)$ is a common boundary fixed point of $\mathcal{S}$ in $\bar{\Delta}$, and no element F_t $(t > 0)$ has an interior fixed point in Δ.

Also, we note that if τ in (1) belongs to $\partial\Delta$, then if follows by a result in [10] that the angular limits,

$$f(\tau) := \angle \lim_{z \to \tau} f(z) = 0 \quad \text{and} \quad f'(\tau) := \angle \lim_{z \to \tau} f'(z) = \beta$$

exist and that β is a nonnegative real number (see also [6]). Moreover, if for some point $\zeta \in \partial\Delta$ there are limits,

$$\angle \lim_{z \to \zeta} f(z) = 0 \quad \text{and} \quad \angle \lim_{z \to \zeta} f'(z) = \gamma,$$

with $\gamma \geqslant 0$, then $\gamma = \beta$ and $\zeta = \tau$ (see [10] and [15]).

In the case where $\beta > 0$, the semigroup $\mathcal{S} = \{F_t\}_{t \geqslant 0}$ consists of mappings $F_t \in \mathrm{Hol}(\Delta)$ of **hyperbolic type,** $\angle \lim_{z \to \tau} \frac{\partial F_t(z)}{\partial z} = e^{-t\beta} < 1$. Otherwise ($\beta = 0$), it consists of mappings of **parabolic type**, $\angle \lim_{z \to \tau} \frac{\partial F_t(z)}{\partial z} = 1$ for all $t \geqslant 0$.

Definition 2. A point $\eta \in \partial\Delta$, is said to be a **boundary regular null point** of $f \in \mathrm{Hol}(D, \mathbb{C})$ if $f(\eta) := \angle \lim_{z \to \eta} f(z) = 0$ and $\gamma = \angle \lim_{z \to \eta} f'(z)$ exists finitely.

It follows by a result in [15] (see also [6]) that if $f \in \mathrm{Hol}(D, \mathbb{C})$ is the generator of a semigroup $\mathcal{S} = \{F_t\}_{t \geqslant 0}$ having a boundary regular null point $\eta \in \partial\Delta$ with $\gamma = \angle \lim_{z \to \eta} f'(z)$, then γ is a real number. Moreover, $\gamma \geqslant 0$ if and only if $\eta \in \partial\Delta$ is the Denjoy–Wolff point of $\mathcal{S}$; otherwise ($\gamma < 0$), η is a repelling fixed point for $\mathcal{S}$.

It turns out that if a semigroup $\mathcal{S}$ generated by $f \in \mathrm{Hol}(D, \mathbb{C})$ contains neither elliptic automorphisms of Δ nor a parabolic type self-mapping of Δ, then the solvability of our problem mentioned above is equivalent to the existence of a boundary regular null point of the generator f different from the Denjoy–Wolff point of $\mathcal{S}$. Actually, more is true.

Definition 3. Let $\mathcal{S} = \{F_t\}_{t \geqslant 0}$ be a semiflow on Δ. A domain $\Omega \subset \Delta$ is called a **backward flow-invariant domain** (shortly, **BFID**) for $\mathcal{S}$ if $\mathcal{S} \subset \mathrm{Aut}(\Omega)$.

Theorem 1. *Let $\mathcal{S} = \{F_t\}_{t \geqslant 0}$ be a nontrivial semiflow on Δ generated by $f \in \mathrm{Hol}(D, \mathbb{C})$ which does not contain an elliptic automorphism of Δ. The following assertions are equivalent*:

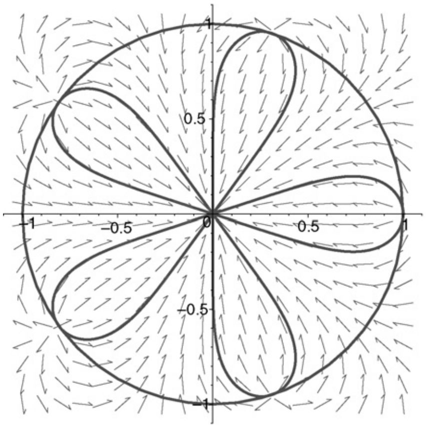

Fig. 1. BFID's for the semigroup generated by $f(z) = z(1 - z^5)$.

(i) *f has a boundary regular null point $\eta \in \partial\Delta$ different from the Denjoy–Wolff point of $\mathcal{S}$, i.e.,*

$$\gamma = \angle \lim_{z\to\eta} f'(z) < 0;$$

(ii) *for some $\alpha > 0$, the differential equation,*

$$\alpha\varphi'(z)(z^2 - 1) = 2f(\varphi(z)), \tag{2}$$

has a locally univalent solution φ with $|\varphi(z)| < 1$ when $z \in \Delta$.

Moreover, in this case, $\alpha \geqslant -\gamma$, φ is univalent and is a Riemann conformal mapping of Δ onto a backward flow invariant domain $\Omega \subset \Delta$, so $\mathcal{S} \subset \mathrm{Aut}(\Omega)$.

The following result contains a partial converse:

Theorem 2. *Let $\mathcal{S} = \{F_t\}_{t\geqslant 0}$ be a semiflow on Δ generated by f, and let $\tau \in \bar{\Delta}$ be its Denjoy–Wolff point with $f(\tau) = 0$ and $f'(\tau) = \beta$, $\operatorname{Re}\beta > 0$. If $\Omega \subset \Delta$ is a nonempty backward flow invariant domain for $\mathcal{S}$, then it is a Jordan domain such that $\tau \in \partial\Omega$, and there is a point $\eta \in \partial\Omega \cap \partial\Delta$ such that $\lim_{t\to-\infty} F_t(z) = \eta$ whenever $z \in \Omega$, $\angle \lim_{z\to\eta} f(z) = 0$ and $\angle \lim_{z\to\eta} f'(z) =: \gamma$ exists with $\gamma < 0$. In addition, there is a conformal mapping φ of Δ onto Ω which satisfies Eq.* (2) *with some $\alpha \geqslant -\gamma$.*

Definition 4. A backward flow invariant domain (BFID) $\Omega \subset \Delta$ for $\mathcal{S}$ is said to be **maximal** if there is no $\Omega_1 \supset \Omega$, $\Omega_1 \neq \Omega$, such that $\mathcal{S} \subset \mathrm{Aut}(\Omega_1)$.

Theorem 3. *Let $\mathcal{S} = \{F_t\}_{t\geqslant 0}$ be a semiflow on Δ generated by f, and let $\eta \in \partial\Delta$ be a boundary regular null point of f with $\gamma = \angle \lim_{z\to\eta} f'(z) < 0$. Let φ be a* (*univalent*) *solution of* (2) *with $\alpha \geqslant -\gamma$ normalized by $\varphi(1) = \tau$ and $\varphi(-1) = \eta$. The following assertions are equivalent*:

(i) *$\Omega = \varphi(\Delta)$ is a maximal BFID*;
(ii) *$\alpha = -\gamma$*;
(iii) *φ is isogonal at the boundary point $z = -1$.*

In general, a maximal BFID for $\mathcal{S}$ need not be unique. Moreover, if a semigroup $\mathcal{S} = \{F_t\}_{t\geqslant 0}$ contains neither elliptic automorphisms of Δ nor a self-mapping of parabolic type, then there is *a one-to-one correspondence between maximal flow invariant domains for $\mathcal{S}$ and repelling fixed points.* This fact determines a flower structure of the collection of BFID's around the Denjoy–Wolff point (see Fig. 1).

Theorem 4. *Let $\mathcal{S} = \{F_t\}_{t\geqslant 0}$ be a semiflow on Δ generated by f. Assume that there is a sequence $\{\eta_k\} \in \partial\Delta$ of boundary regular null points of f, i.e., $f(\eta_k) = 0$ and $\gamma_k = f'(\eta_k) > -\infty$. Then the following assertions hold.*

(i) *There is* $\delta > 0$ *such that* $\gamma_k < -\delta < 0$ *for all* $k = 1, 2, \ldots$.
(ii) *For each* $a < -\delta < 0$ *there is at most a finite number of the points* η_k *such that* $a \leqslant \gamma_k < -\delta$. *Consequently, Eq.* (2) *has a* (*univalent*) *solution* $\varphi \in \mathrm{Hol}(\Delta)$ *for each* $\alpha \geqslant -\max\{\gamma_k\} > -\delta$.
(iii) *If* φ_k *is a solution of* (2) *normalized by* $\varphi_k(1) = \tau$, $\varphi_k(-1) = \eta_k$ *with* $\alpha = \gamma_k$ *and* $\Omega_k = \varphi_k(\Delta)$ (*i.e.,* Ω_k *are maximal*)*, then for each pair* Ω_{k_1} *and* Ω_{k_2} *such that* $\eta_{k_1} \neq \eta_{k_2}$ *either* $\overline{\Omega_{k_1}} \cap \overline{\Omega_{k_2}} = \{\tau\}$ *or* $\overline{\Omega_{k_1}} \cap \overline{\Omega_{k_2}} = l$*, where* l *is a continuous curve joining* τ *with a point on* $\partial\Delta$.

The proofs of our theorems are based on linearization models for semigroups constructed by solutions of Schröder's and Abel's functional equations (see, for example, [12,3,7,8] and [5]). The main tools in the study of geometric properties of these solutions are recent developments in the theory of starlike and spirallike functions with respect to a boundary point (see [14,11,16,9] and [2]). On the way to solving these problems, we prove a new angle distortion theorem for starlike and spiral-like functions with respect to interior and boundary points.

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