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Mathematical Analysis/Probability Theory

On multifractal time subordination

Stéphane Seuret

Laboratoire d'analyse et de mathématiques appliquées, UFR sciences et technologie, Université Paris-Est, 61, avenue du Général-de-Gaulle, 94010 Créteil cedex, France

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Abstract

Let $Z : [0, 1] \to \mathbb{R}$ be a continuous function. We are interested in the existence of a decomposition of Z as $Z = g \circ f$, where $g : [0, 1] \to \mathbb{R}$ is a monofractal function with exponent $0 < H < 1$ and $f : [0, 1] \to [0, 1]$ is a time subordinator (i.e. f is the integral of a positive Borel measure supported by [0, 1]). We prove that such a decomposition can be found for a large class of functions, and when the initial function Z is given, the monofractality exponent of the associated function g is uniquely determined. The assumptions made on Z are weak and rather natural.

This yields new insights in the understanding of multifractal behaviors of functions, giving an important role to the regularity analysis of Borel measures. This work also allows us to find a very interesting relationship between self-similar functions and self-similar measures. ***To cite this article: S. Seuret, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Sur le changement de temps multifractal. Soit $Z : [0, 1] \to \mathbb{R}$ une fonction continue. Nous prouvons que pour toute une classe de fonctions continues Z ayant un comportement d'échelle homogène (en un sens que nous précisons), la fonction Z peut se décomposer en $Z = g \circ f$, où g est une fonction monofractale d'indice $0 < H < 1$ et f est un homéomorphisme croissant de [0, 1]. Ainsi, f étant un subordinateur en temps, Z s'interprète comme la fonction monofractale g en temps f. Nous expliquons pourquoi les conditions imposées à Z sont faibles, et avant tout naturelles.

Ce résultat permet de mieux comprendre les comportements locaux des fonctions continues. En effet, les variations locales de continuité sont essentiellement dûes aux variations de continuité de f, qui peut être vue comme l'intégrale d'une mesure borélienne. Ceci donne une importance prépondérante à l'analyse de régularité locale des mesures boréliennes par rapport à l'analyse des fonctions. Ce travail permet également de relier de façon satisfaisante les mesures auto-similaires aux fonctions auto-similaires via un changement de temps. ***Pour citer cet article : S. Seuret, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

E-mail address: seuret@univ-paris12.fr.

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Étant donnée $Z:[0,1]\to\mathbb{R}$ une fonction continue, nous cherchons des conditions sous lesquelles elle peut se décomposer sous la forme $Z=g\circ f$, où $g:[0,1]\to\mathbb{R}$ est une fonction monofractale d'exposant $0<H<1$ et $f:[0,1]\to[0,1]$ est un homéomorphisme croissant.

Rappelons que la régularité locale d'une fonction localement bornée est quantifiée grâce à l'exposant ponctuel de Hölder, défini comme suit : Si $Z\in L^{\infty}_{\mathrm{loc}}([0,1])$, $\alpha\geqslant 0$ et $t_0\in[0,1]$, Z appartient à $C^{\alpha}_{t_0}$ si l'on peut trouver un polynôme P de degré plus petit que $[\alpha]$ et une constante C tels que

$$\text{pour tout } t \text{ suffisamment proche de } t_0, \text{ on a } \left|Z(t)-P(t-t_0)\right|\leqslant C|t-t_0|^{\alpha}. \tag{1}$$

L'exposant ponctuel de Hölder de Z en t_0 vaut alors $h_Z(t_0)=\sup\{\alpha\geqslant 0:\ f\in C^{\alpha}_{t_0}\}$, et le spectre de singularité de Z est l'application $d_Z(h)=\dim\{t:\ h_Z(t)=h\}$ (dim représente ici la dimension de Hausdorff, avec par convention $\dim\emptyset=-\infty$).

Une fonction monofractale d'exposant $H>0$ a en chaque temps t le même exposant ponctuel, et son spectre de singularité est réduit à un point : $d_F(H)=1$ et $d_F(h)=-\infty$ si $h\neq H$. Les fonctions de Weierstrass ou (presque surement) les trajectoires de mouvement Brownien (fractionnaire ou non) fournissent des exemples de fonctions monofractales. Les fonctions multifractales ont généralement un spectre de singularité dont le support est un intervalle non trivial. De nombreux phénomènes physiques sont maintenant connus pour engendrer des signaux à propriétés multifractales, c'est-à-dire que leur régularité locale peut varier de manière très erratique d'un temps t à un autre.

Un procédé efficace pour créer des processus multifractals à partir de fonctions monofractales est de les subordonner en temps, c'est-à-dire de les composer avec une fonction $f:[0,1]\to[0,1]$ strictement croissante. Mandelbrot [5] a montré l'intérêt de cette approche dans le contexte de l'analyse financière (il composait des processus avec des intégrales de mesures multifractales). La question naturelle est de savoir si les processus ainsi créés constituent un sous-ensemble important des fonctions continues sur [0, 1].

Nous allons montrer que toute fonction Z ayant un comportement d'échelle « homogène » (i.e. satisfaisant aux conditions du Théorème 0.1 ci-dessous) peut s'écrire comme composition d'une fonction monofractale avec un subordinateur en temps.

Auparavant, rappelons que l'oscillation de Z sur un intervalle dyadique $I_{j,k}=[k2^{-j},(k+1)2^{-j})$, $j\geqslant 1$ et $k\in\{0,1,\ldots,2^j-1\}$, se définit par $\omega_{j,k}(Z)=\sup_{t,t'\in I_{j,k}}|Z(t)-Z(t')|=\sup_{t\in I_{j,k}}Z(t)-\inf_{t\in I_{j,k}}Z(t)$. Considérons alors pour tout $j\geqslant 1$ l'unique nombre réel $H_j(Z)$ et l'exposant $H(Z)$ définis par

$$\sum_{k=0}^{2^j-1}\left(\omega_{j,k}(Z)\right)^{1/H_j(Z)}=1 \quad \text{et} \quad H(Z)=\liminf_{j\to+\infty}H_j(Z).$$

Enfin, pour tout $J\geqslant 0$ et $K\in\{0,\ldots,2^j-1\}$, posons $Z_{J,K}:t\in[0,1]\longmapsto\frac{Z\circ\varphi_{J,K}(t)}{\omega_{J,K}(Z)}\in\mathbb{R}$, où $\varphi_{J,K}$ est la transformation affine canonique qui envoie [0, 1] sur $I_{J,K}$. Nous montrons le théorème suivant :

Théorème 0.1. *Soit $Z:[0,1]\to\mathbb{R}$ une fonction continue. Supposons que*

(i) *Il existe $0<H<1$ tel que pour tout $J\geqslant 0$ et $K\in\{0,\ldots,2^j-1\}$, $H(Z_{J,K})=H(=H(Z))$,*
(ii) *Il existe deux suites $(\varepsilon_J)_{J\geqslant 0}$ et $(\eta_J)_{J\geqslant 0}$, $\kappa>0$ et deux réels $0<\alpha<\beta$ tels que :*
  (a) *$(\varepsilon_J)_{J\geqslant 0}$ et $(\eta_J)_{J\geqslant 0}$ sont deux suites positives décroissantes vers 0, et $\varepsilon_J=\mathrm{o}(1/(\log J)^{2+\kappa})$,*
  (b) *pour tout $J\geqslant 0$ et $K\in\{0,\ldots,2^J-1\}$, la suite $(H_j(Z_{J,K}))_{j\geqslant 1}$ converge vers $H=H(Z_{J,K})$ à la vitesse suivante : pour tout $j\geqslant[J\eta_J]$, $|H-H_j(Z_{J,K})|\leqslant\varepsilon_J$,*
  (c) *pour tout $J\geqslant 0$ et $K\in\{0,\ldots,2^J-1\}$, on a que $\forall j\geqslant[J\eta_J]$, $\forall k\in\{0,\ldots,2^j-1\}$, $2^{-j\beta}\leqslant\omega_{j,k}(Z_{J,K})\leqslant 2^{-j\alpha}$.*

Alors il existe une fonction monofractale g d'indice $H(Z)$ et un homéomorphisme f croissant de [0, 1] *tels que $Z=g\circ f$.*

Les conditions imposées à Z sont naturelles : Si la condition (i) n'est pas vérifiée, alors la fonction Z ne peut pas se décomposer (voir Lemme 1.5 ci-après). Les conditions (ii)(a) et (ii)(b) garantissent l'existence d'une décomposition, et sont réalisées par les grandes classes d'objets monofractals et multifractals. La dernière condition est très raisonnable pour une fonction qui a une régularité Höldérienne globale minimale.

Les principaux objets multifractals sont les fonctions et les mesures dites auto-similaires. La fonction $Z : [0,1] \to [0,1]$ est auto-similaire si l'on peut trouver d similitudes $S_0, S_1, \ldots, S_{d-1}$ de $[0,1]$ vers $[0,1]$, de rapport respectivement $r_0, r_1, \ldots, r_{d-1}$, et $(\lambda_0, \lambda_1, \ldots, \lambda_{d-1}) \in \mathbb{R}^d$ tels que

$$\forall t \in [0,1], \quad Z(t) = \sum_{k=0}^{d-1} \lambda_k \cdot \left(Z \circ (S_k)^{-1}\right)(t) + \phi(t).$$

On supposera la condition (6) satisfaite. Considérons alors l'unique $\beta > 1$ tel que $\sum_{k=0}^{d-1} |\lambda_k|^\beta = 1$, et le vecteur de probabilités associé $(p_0, p_1, \ldots, p_{d-1}) = (|\lambda_0|^\beta, |\lambda_1|^\beta, \ldots, |\lambda_{d-1}|^\beta)$. Il existe alors une unique mesure dite auto-similaire définie par l'équation $\mu = \sum_{k=0}^{d-1} p_k \cdot (\mu \circ S_k^{-1})$.

Nous trouvons une relation frappante entre la fonction auto-similaire Z et la mesure auto-similaire qui lui est naturellement associée.

Théorème 0.2. *La relation* (6) *assure que Z existe et est unique* [3]. *Alors soit Z est κ-Lipschitz, pour une certaine constante $\kappa > 0$, soit Z est décomposable et il existe une fonction monofractale g d'exposant $1/\beta$ tel que pour tout $t \in [0,1]$, $Z(t) = g(\mu[0,t])$.*

1. Introduction

Local regularity and multifractal analysis have become unavoidable issues in the past years. Indeed, physical phenomena exhibiting wild local regularity properties have been discovered in many contexts (turbulence flows, intensity of seismic waves, traffic analysis, ...). From a mathematical viewpoint, the multifractal approach is also a fruitful source of interesting problems. Consequently, there is a strong need for a better theoretical understanding of the so-called multifractal behaviors. In this note, we exhibit some relations between multifractal properties and time subordination for continuous functions.

Let us recall how the local regularity of a locally bounded function is measured:

Definition 1.1. Let $Z \in L^\infty_{\text{loc}}([0,1])$. For $\alpha \geqslant 0$ and $t_0 \in [0,1]$, Z is said to belong to $C^\alpha_{t_0}$ if there are a polynomial P of degree less than $[\alpha]$ and a constant C such that, locally around t_0, $|Z(t) - P(t - t_0)| \leqslant C|t - t_0|^\alpha$. The pointwise Hölder exponent of Z at t_0 is $h_Z(t_0) = \sup\{\alpha \geqslant 0\colon\ f \in C^\alpha_{t_0}\}$.

The singularity spectrum of Z is then defined by $d_Z(h) = \dim\{t\colon\ h_Z(t) = h\}$ (dim stands for the Hausdorff dimension, and $\dim \emptyset = -\infty$ by convention).

A function $Z : [0,1] \to \mathbb{R}$ is monofractal with exponent $0 < H < 1$ when $h_Z(t) = H$ for every $t \in [0,1]$. For monofractal functions Z, $d_Z(H) = 1$, while $d_Z(h) = -\infty$ for $h \neq H$. Sample paths of Brownian motions or fractional Brownian motions are known to be almost surely monofractal. Starting from a monofractal process as above in dimension 1, an efficient way to get a more elaborate process is to compose it with a time subordinator, i.e. an increasing function or process. Mandelbrot showed the relevance of time subordination in the study of financial data [5]. From a theoretical viewpoint, it is also challenging to understand how the multifractal properties of a function are modified after a time change [6,1].

A natural question is to understand the differences between multifractal processes and compositions of monofractal functions with multifractal subordinators.

Definition 1.2. A function $Z : [0,1] \to \mathbb{R}$ is said to be the composition of a monofractal function with a time subordinator (CMT) when Z can be written as

$$Z = g \circ f, \tag{2}$$

where $g : [0,1] \to \mathbb{R}$ is monofractal with exponent $0 < H < 1$ and $f : [0,1] \to [0,1]$ is an increasing homeomorphism of $[0,1]$.

Let us begin with two cases where a function Z is obviously CMT:

1. If Z is the integral of any positive Borel measure μ, then $Z = Id_{[0,1]} \circ Z$, where the identity $Id_{[0,1]}$ is monofractal and Z is increasing.

2. Any monofractal function Z_H can be written $Z_H = Z_H \circ Id_{[0,1]}$, where Z_H is monofractal and $Id_{[0,1]}$ is undoubtably an homeomorphism of $[0,1]$.

To bring general answers to our problem and thus to exhibit interesting CMT functions, we develop an approach based on the oscillations of a function $Z : [0,1] \to \mathbb{R}$. For every subinterval $I \subset [0,1]$, consider the oscillations of order 1 of Z on I defined by

$$\omega_I(Z) = \sup_{t,t' \in I} \left| Z(t) - Z(t') \right| = \sup_{t \in I} Z(t) - \inf_{t \in I} Z(t).$$

We assume that Z is continuous and for every non-trivial subinterval I of $[0,1]$, $\omega_I(Z) > 0$. This entails that Z is nowhere locally constant, which is a natural assumption for the results we are looking for. The oscillations of order 1 characterize precisely the pointwise Hölder exponents strictly less than 1 [4].

Lemma 1.3. *Let $Z : [0,1] \to \mathbb{R}$ a C^γ function, for some $\gamma > 0$. Assume that $h_Z(t) < 1$. Then*

$$h_Z(t) = \liminf_{r \to 0^+} \frac{|\log \omega_{B(t,r)}(Z)|}{|\log r|} = \liminf_{j \to +\infty} \frac{|\log_2 \omega_{B(t,2^{-j})}(Z)|}{j}.$$

Let us introduce the quantity that will be the basis of our construction.

For every $j \geqslant 1$, $k \in \{0, \ldots, 2^j - 1\}$, we consider the dyadic intervals $I_{j,k} = [k2^{-j}, (k+1)2^{-j})$, so that $\bigcup_{k=0,\ldots,2^j-1} I_{j,k} = [0,1[$, the union being disjoint. For every $j \geqslant 1$ and $k \in \{0, \ldots, 2^j - 1\}$, for simplicity we set $\omega_{j,k}(Z) = \omega_{I_{j,k}}(Z)$ $(= \omega_{\overline{I_{j,k}}}(Z)$ since Z is $C^0)$.

Definition 1.4. For every $j \geqslant 1$, let $H_j(Z)$ be the unique real number such that $\sum_{k=0}^{2^j-1} (\omega_{j,k}(Z))^{1/H_j(Z)} = 1$. We then define the intrinsic monofractal exponent $H(Z)$ of Z as $H(Z) = \liminf_{j \to +\infty} H_j(Z)$.

This quantity $H(Z)$ characterizes the asymptotic maximal values of the oscillations of Z on the whole interval $[0,1]$. This exponent is the core of our theorem, because it gives an upper limit to the maximal time distortions we are allowed to apply. In addition, $H(Z)$ has a functional interpretation. Indeed, if Z can be decomposed as (2), then the exponent of the monofractal function g does not depend on the oscillation approach nor on the dyadic basis. One can see that when $H(Z) \leqslant 1$,

$$H(Z) = \inf\left\{p > 0\colon\ Z \in B^{1/p,\infty}_{p,\mathrm{loc}}\big((0,1)\big)\right\} = \inf\left\{p > 0\colon\ Z \in \mathcal{O}^{1/p}_p\big((0,1)\big)\right\}, \tag{3}$$

where $B^{q,\infty}_{1/q,\mathrm{loc}}((0,1))$ and $\mathcal{O}^{1/p}_p((0,1))$ are respectively the Besov space and *oscillation space* on the open interval $(0,1)$ (see Jaffard in [4] for instance). For multifractal functions Z satisfying some multifractal formalism, the exponent $H(Z)$ can also be read on the singularity spectrum of Z. Indeed, $H(Z)$ corresponds to the inverse of the largest possible slope of a straight line going through 0 and tangent to the singularity spectrum d_Z of Z. These remarks provide us with an idea *a priori* of the monofractal exponent of g in the decomposition $Z = g \circ f$ and with an intrinsic formula for $H(Z)$.

Let us come back to the two simple examples above:

1. For the integral Z of any positive measure μ, $\sum_{k=0}^{2^j-1} \omega_{j,k}(Z) = \sum_{k=0}^{2^j-1} \mu(I_{j,k}) = 1$, hence $H_j(Z) = H(Z) = 1$, which corresponds to the monofractal exponent of $Id_{[0,1]}$ from the oscillations viewpoint.

2. The first difficulties arise for the monofractal functions Z_H. When Z_H is monofractal of exponent H, then we do not have necessarily $H(Z_H) = H$. Nevertheless this holds true for the Weierstrass functions and the sample paths of (fractional) Brownian motions.

The sole knowledge of $H(Z)$ is not sufficient to get relevant results. For instance, consider a function Z that has two different monofractal behaviors on $[0,1/2]$ and $[1/2,1]$. Such a Z can be obtained as the continuous juxtaposition of two Weierstrass functions with distinct exponents $H_1 < H_2$: We have $H(Z) = H_1$, and Z cannot be written as the composition of a monofractal function with a time subordinator. This is a consequence of the following lemma:

Lemma 1.5. *Let g_1 and g_2 be two real monofractal functions on* $[0,1]$ *of distinct exponents* $0 < H_1 < H_2 < 1$. *There is no continuous strictly increasing function* $f:[0,1]\to[0,1]$ *such that* $g_1 = g_2 \circ f$.

Indeed, such a subordinator f would 'dilate' time everywhere, which is impossible. Hence, with a CMT function Z is associated a unique monofractal exponent.

We need to introduce a homogeneity condition C1 to get rid of these annoying and artificial cases. This condition heuristically imposes that the oscillations of any restriction of Z to a subinterval of $[0,1]$ have the same asymptotic properties as the oscillations of Z on $[0,1]$.

Definition 1.6. *Condition* C1: Let $Z:[0,1]\to\mathbb{R}$ be a continuous function. Let $J \geqslant 0$, and $K \in \{0,\dots,2^j-1\}$. Consider $Z_{J,K}$, the function defined as

$$Z_{J,K}: t\in[0,1] \longmapsto \frac{Z\circ\varphi_{J,K}(t)}{\omega_{J,K}(Z)} \in \mathbb{R},$$

where $\varphi_{J,K}$ is the canonical affine contraction which maps $[0,1]$ to $I_{J,K}$. Condition C1 is satisfied for Z when there is $0<H<1$ such that for every $J\geqslant 0$ and $K\in\{0,\dots,2^j-1\}$, $H(Z_{J,K}) = H(=H(Z))$.

The function $Z_{J,K}$ is a renormalized version of the restriction of Z to the interval $I_{J,K}$. Although self-similar functions are good candidates to satisfy C1, a function Z fulfilling this condition does not need at all to enjoy such a property. In order to get a result, we strengthen the convergence toward $H(Z_{J,K})$.

Definition 1.7. *Condition* C2: Assume that C1 is fulfilled. Condition C2 is verified by Z when there are two sequences $(\varepsilon_J)_{J\geqslant 0}$ and $(\eta_J)_{J\geqslant 0}$, $\kappa > 0$ and two real numbers $0<\alpha<\beta$ with the properties:

(i) $(\varepsilon_J)_{J\geqslant 0}$ and $(\eta_J)_{J\geqslant 0}$ are positive non-increasing sequences that converge to 0, and $\varepsilon_J = \mathrm{o}(1/(\log J)^{2+\kappa})$.

(ii) For every $J\geqslant 0$ and $K\in\{0,\dots,2^J-1\}$, the sequence $(H_j(Z_{J,K}))_{j\geqslant 1}$ converges to $H = H(Z_{J,K})$ (it is not only a lim inf, it is a limit) with the following convergence rate:

$$\text{For every } j \geqslant [J\eta_J], \quad \left|H - H_j(Z_{J,K})\right| \leqslant \varepsilon_J. \tag{4}$$

(iii) For every $J\geqslant 0$ and $K\in\{0,\dots,2^J-1\}$, the following regularity condition holds:

$$\text{For every } j\geqslant[J\eta_J],\ \text{for every } k\in\{0,\dots,2^j-1\},\quad 2^{-j\beta}\leqslant \omega_{j,k}(Z_{J,K}) \leqslant 2^{-j\alpha}. \tag{5}$$

Assuming that $H(Z_{J,K})$ is a limit is of course restrictive, but not limiting in practice, since this condition holds true for most of the interesting functions or (almost surely) for most of the sample paths of processes. Similarly, the decreasing behavior (5) is expected for a C^γ function. The convergence speed (4) is an important constraint, but the convergence rate we impose on $(\varepsilon_J)_{J\geqslant 0}$ toward 0 is extremely slow.

Theorem 1.8. *Let* $Z:[0,1]\to\mathbb{R}$ *be a continuous function. Assume that Z satisfies* C1 *and* C2.
Then Z is CMT and the function g in (2) *is monofractal with exponent* $H(Z)$.

Such a decomposition is of course not unique: If Z is CMT and $w:[0,1]\to[0,1]$ is C^∞ and strictly increasing, then $Z = (g\circ w)\circ(w^{-1}\circ f)$, where $g\circ w$ is still a monofractal function of exponent $H(Z)<1$ and $w^{-1}\circ f$ is an increasing function. Nevertheless, if two decompositions (2) exist respectively with functions g_1, g_2, f_1 and f_2, then g_1 and g_2 are necessarily monofractal with the same exponent $H(Z)$. This is again a consequence of Lemma 1.5.

An important consequence of Theorem 1.8 is that the (possibly) multifractal behavior of Z is contained in the multifractal behavior of f. Indeed, since g is monofractal with exponent H, the variations of the local regularity of Z are necessarily due to some variations of regularity for f. Theoretically, Theorem 1.8 increases the role of the multifractal analysis of measures, since for the functions satisfying C1 and C2, their multifractal behavior is ruled exclusively by the behavior of μ.

In the same spirit as Theorem 1.8, Theorem 1.9 relates the so-called self-similar functions Z introduced in [3] with the self-similar measures naturally associated with Z.

Let ϕ be a Lipschitz function on [0, 1] (we suppose that the Lipschitz constant C_ϕ equals 1), and let $S_0, S_1, \ldots, S_{d-1}$ be d contractive non-trivial similitudes satisfying:

(i) for every $i \neq j$, $S_i((0,1)) \cap S_j((0,1)) = \emptyset$ (open set condition),
(ii) $\bigcup_{i=0}^{d-1} S_i([0,1]) = [0,1]$ (the intervals $S_i([0,1])$ form a covering of [0, 1]).

We denote by $0 < r_0, r_1, \ldots, r_{d-1} < 1$ the ratios of the similitudes $S_0, \ldots, S_{d-1}$. By construction $\sum_{k=0}^{d-1} r_k = 1$. Let $\lambda_0, \lambda_1, \ldots, \lambda_{d-1}$ be d non-zero real numbers, which satisfy

$$0 < \chi_{\min} = \min_{k=0,\ldots,d-1} \left|\frac{r_k}{\lambda_k}\right| \leqslant \chi_{\max} = \max_{k=0,\ldots,d-1} \left|\frac{r_k}{\lambda_k}\right| < 1. \tag{6}$$

A function $Z : [0,1] \to [0,1]$ is called self-similar when Z satisfies the following functional equation

$$\forall t \in [0,1], \quad Z(t) = \sum_{k=0}^{d-1} \lambda_k \cdot \big(Z \circ (S_k)^{-1}\big)(t) + \phi(t). \tag{7}$$

Relation (6) ensures that Z exists and is unique [3].

Let us consider the unique exponent $\beta > 1$ such that $\sum_{k=0}^{d-1} |\lambda_k|^\beta = 1$. This β is indeed greater than 1, since $\sum_{k=0}^{d-1} r_k = 1$ and $|\lambda_k| > r_k$ for all k by (6). With the probability vector $(p_0, p_1, \ldots, p_{d-1}) = (|\lambda_0|^\beta, |\lambda_1|^\beta, \ldots, |\lambda_{d-1}|^\beta)$ and the similitudes $(S_k)_{k=0,\ldots,d-1}$ can be associated the unique self-similar probability measure μ satisfying

$$\mu = \sum_{k=0}^{d-1} p_k \cdot \big(\mu \circ S_k^{-1}\big). \tag{8}$$

The function Z (7) and the measure μ (8) are two of the most classical multifractal objects.

Theorem 1.9. *Let Z and μ be defined by* (7) *and* (8)*. Then, either Z is a κ-Lipschitz function for some constant $\kappa > 0$ or Z is CMT and there is a monofractal function g of exponent $1/\beta$ such that for every $t \in [0,1]$, $Z(t) = g(\mu[0,t])$.*

Theorem 1.9 establishes a very satisfactory relationship between self-similar functions and self-similar measures, that somehow was expected. It also asserts that the choice of the function ϕ influences only the monofractal function g, not the time subordinator. The multifractal analysis of Z follows from the multifractal analysis of μ, which is a very classical problem (see [2]). This also illustrates our purpose: in this case, the local regularity analysis of the functions we consider reduces to the local regularity analysis of some Borel measure μ.

The proofs of Theorems 1.8 and 1.9 are constructive and can be found in [7].

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