



Partial Differential Equations

# Solutions to the nonlinear Schrödinger equation carrying momentum along a curve

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## Abstract

We study the nonlinear Schrödinger equation  $-\varepsilon^2 \Delta \psi + V(x)\psi = |\psi|^{p-1}\psi$  on a compact manifold or on  $\mathbb{R}^n$ , where  $V$  is a positive potential and  $p > 1$ . As  $\varepsilon$  tends to zero, we prove existence of complex-valued solutions which concentrate along closed curves and whose phase is highly oscillatory, carrying quantum-mechanical momentum along the limit set. **To cite this article:** *F. Mahmoudi et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).*

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## Résumé

**Solutions d'une équation de Schrödinger non linéaire portant un moment le long d'une courbe.** On étudie l'équation de Schrödinger non linéaire  $-\varepsilon^2 \Delta \psi + V(x)\psi = |\psi|^{p-1}\psi$  sur une variété compacte ou sur  $\mathbb{R}^n$ , où  $V$  est un potentiel positif, régulier et  $p > 1$ . Lorsque  $\varepsilon$  tend vers zéro, on montre l'existence de solutions à valeurs complexes qui se concentrent le long d'une courbe fermée et dont la phase est hautement oscillante, portant un moment quantique le long de l'ensemble limite. **Pour citer cet article :** *F. Mahmoudi et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).*

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## Version française abrégée

On montre l'existence d'une classe spéciale de solutions d'une équation de Schrödinger (elliptique) non linéaire  $-\varepsilon^2 \Delta \psi + V(x)\psi = |\psi|^{p-1}\psi$ , sur une variété compacte ou sur l'espace euclidien. Ici  $V$  représente le potentiel,  $p$  un exposant plus grand que 1 et  $\varepsilon$  un petit paramètre qui correspond à la constante de Planck. Lorsque  $\varepsilon$  tend vers zéro (i.e. dans le cas de la limite semi classique) on montre l'existence de solutions à valeurs complexes qui se concentrent le long d'une courbe fermée, et dont la phase est hautement oscillante. Physiquement, ces solutions portent un moment quantique le long de la courbe limite. Ce résultat répond à une conjecture énoncée dans [2] pour le cas où l'ensemble limite est unidimensionnel et prolonge un résultat précédent dans [4] (où des solutions à valeurs réelles pour  $n = 2$  ont été construites). L'idée de la démonstration est de construire dans un premier temps une bonne solution approchée

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$\tilde{\psi}$  en utilisant la stationnarité et la non-dégénérescence de la courbe, les vraies solutions peuvent alors être trouvées en perturbant  $\tilde{\psi}$  et en utilisant un théorème de point fixe pour les applications contractantes. On montre aussi que l'opérateur linéarisé au voisinage de  $\tilde{\psi}$  admet un noyau non trivial dû à l'invariance de l'équation par translation et par rotation complexe, ceci donne lieu à un phénomène de résonance. Pour pouvoir s'affranchir de cette difficulté, on utilise une méthode basée sur une réduction de Lyapunov–Schmidt (le paramètre  $\varepsilon$  est choisi de telle sorte qu'il évite certaines valeurs critiques).

## 1. Introduction

In this Note we are concerned with concentration phenomena for solutions to the singularly-perturbed elliptic problem:

$$-\varepsilon^2 \Delta_g \psi + V(x)\psi = |\psi|^{p-1}\psi \quad \text{on } M, \quad (\text{NLS}_\varepsilon)$$

where  $M$  is an  $n$ -dimensional compact manifold (or the flat Euclidean space  $\mathbb{R}^n$ ),  $V$  a smooth positive function on  $M$  satisfying  $0 < V_1 \leq V \leq V_2$ ;  $\|V\|_{C^3} \leq V_3$ ,  $\psi$  a complex-valued function,  $\varepsilon > 0$  a small parameter,  $p$  an exponent greater than 1 and where  $\Delta_g$  stands for the Laplace–Beltrami operator on  $M$ .

(NLS $_\varepsilon$ ) arises from the study of the Nonlinear Schrödinger Equation:

$$i\hbar \partial \tilde{\psi} / \partial t = -\hbar^2 \Delta \tilde{\psi} + V(x)\tilde{\psi} - |\tilde{\psi}|^{p-1}\tilde{\psi} \quad \text{on } M \times [0, +\infty), \quad (1)$$

where  $\tilde{\psi} = \tilde{\psi}(x, t)$  is the *wave function*,  $V$  a potential, and  $\hbar$  the *Planck constant*. A special class of solutions to (1) is constituted by the functions whose dependence on the variables  $x$  and  $t$  is of the form  $\tilde{\psi}(x, t) = e^{-i\frac{\omega t}{\hbar}} \psi(x)$ . Such solutions are called *standing waves* and up to a substitution of  $V(x)$  by  $V(x) - \omega$ , they satisfy (NLS $_\varepsilon$ ), for  $\varepsilon = \hbar$ . An interesting case is the *semiclassical limit*  $\varepsilon \rightarrow 0$ , where one should expect to recover the Newton law of classical mechanics. In particular, near stationary points of the potential, one is lead to search highly concentrated solutions, which could mimic point-particles at rest. There is an extensive literature about this topic, and for brevity we refer the interested readers to Chapter 8 in [1] and the references therein. However it has been conjectured for some time that (NLS $_\varepsilon$ ) might exhibit solutions concentrating along curves or in general higher-dimensional sets (see [14] and [13] for related issues). Recently some answers have been given in [2] and [4] (see also references therein) concerning existence of solutions whose limit profile, once scaled properly in  $\varepsilon$ , is independent of some of the variables. In this Note, and in more detail in [7,8], we consider solutions with oscillatory phase, concentrating along curves and carrying quantum-mechanical momentum along the limit set.

## 2. Main results

Our aim is to construct a new type of solutions to (NLS $_\varepsilon$ ), which concentrate along a curve  $\gamma$  in  $M$ , and whose phase is highly oscillatory. These functions are complex-valued and their profile (after a scaling in  $\varepsilon$ ) near any point  $x_0$  in the image of  $\gamma$ , is asymptotic to a solution of,

$$-\Delta \phi + V(x_0)\phi = |\phi|^{p-1}\phi \quad \text{in } \mathbb{R}^n, \quad (2)$$

decaying exponentially to zero away from the  $x_n$  axis and periodic in  $x_n$ . More precisely, we take,

$$\phi(x', x_n) = e^{-i\hat{f}x_n} \hat{U}(x'), \quad x' = (x_1, \dots, x_{n-1}),$$

where  $\hat{f}$  is a constant and  $\hat{U}(x')$  a real function. With this choice of  $\phi$ , the function  $\hat{U}$  satisfies the equation:

$$-\Delta \hat{U} + (\hat{f}^2 + V(x_0))\hat{U} = |\hat{U}|^{p-1}\hat{U} \quad \text{in } \mathbb{R}^{n-1}, \quad (3)$$

and decays to zero at infinity. Solutions to (3) can be found by considering the (real) function  $U \in C_0(\mathbb{R}^{n-1})$  satisfying  $-\Delta U + U = U^p$  in  $\mathbb{R}^{n-1}$  (which is known to exist if  $p < \frac{n+1}{n-3}$ ), and using the scaling

$$\hat{U}(x') = \hat{h}U(\hat{k}x'), \quad \hat{h} = (\hat{f}^2 + V(x_0))^{\frac{1}{p-1}}, \quad \hat{k} = (\hat{f}^2 + V(x_0))^{\frac{1}{2}}. \quad (4)$$

In the above formulas the constant  $\hat{f}$  can be taken arbitrarily, while  $\hat{h}, \hat{k}$  have to be chosen accordingly, depending on  $V(x_0)$ . In fact  $\hat{f}$  is proportional to the speed of the phase oscillation, and is physically related to the velocity of the quantum-mechanical particle represented by the wave function.

We are aware of only one paper in this direction, [3], where the case of an axially-symmetric potential is treated, and solutions are found via separation of variables: our goal here is to treat this phenomenon in a generic situation, without any symmetry restriction. We also refer to [14] for some related issues.

If we look for a solution  $\psi_\varepsilon$  to  $(\text{NLS}_\varepsilon)$  with this profile, then this should qualitatively behave like:

$$\psi_\varepsilon(\bar{s}, \zeta) = e^{-i \frac{f(\bar{s})}{\varepsilon}} h(\bar{s}) U(k(\bar{s})\zeta/\varepsilon), \tag{5}$$

where  $\bar{s}$  stands for the arc-length parameter of  $\gamma$ , and  $\zeta$  for a system of normal geodesic coordinates. For having more flexibility, we chose the speed of phase oscillation  $f(\bar{s})$  to depend on the point  $\gamma(\bar{s})$ , while  $h(\bar{s}), k(\bar{s})$  should satisfy a formula similar to (4) (with obvious modifications).

The function  $f(\bar{s})$  can be determined using an expansion of  $(\text{NLS}_\varepsilon)$  at order  $\varepsilon$ : one can show that

$$f'(\bar{s}) \simeq \mathcal{A} h^\sigma(\bar{s}) \quad \text{with } \sigma = (n - 1)(p - 1)/2 - 2, \tag{6}$$

where  $\mathcal{A}$  is a constant to be specified. At this point, only the curve  $\gamma$  has to be determined. First, we notice that the phase should be a periodic function in the length of the curve, and therefore it is natural to work in the class of loops  $\Gamma := \{\gamma : \mathbb{R} \rightarrow M, \text{ periodic: } \mathcal{A} \int_\gamma h(\bar{s})^\sigma d\bar{s} = \text{constant}\}$ . Problem  $(\text{NLS}_\varepsilon)$  has a variational structure, with Euler functional  $E_\varepsilon(\psi) = \frac{1}{2} \int_M (\varepsilon^2 |\nabla_g \psi|^2 + V(x) |\psi|^2) - \frac{1}{p+1} \int_M |\psi|^{p+1}$ . Making the *ansatz* (5), by a scaling argument one finds:

$$E_\varepsilon(\psi) \simeq \varepsilon^{n-1} \int_\gamma h(\bar{s})^\theta d\bar{s}, \quad \text{with } \theta = p + 1 - (p - 1)(n - 1)/2, \tag{7}$$

therefore a candidate limit curve  $\gamma$  should be a critical point of the functional  $\int_\gamma h(\bar{s})^\theta d\bar{s}$  restricted to  $\Gamma$ . One can check that the extremality condition is the following:

$$\nabla^N V = \mathbf{H}(h^{p-1}(p - 1)/\theta - 2\mathcal{A}^2 h^{2\sigma}), \tag{8}$$

where  $\nabla^N V$  represents the normal gradient of  $V$  and  $\mathbf{H}$  the curvature of  $\gamma$ . By a long but straightforward calculation, one can find a natural nondegeneracy condition for stationary points, which is expressed by the invertibility of the operator:

$$\begin{aligned} \mathcal{V} \mapsto & - \left( h^\theta - \frac{2\mathcal{A}^2 \theta}{p-1} h^\sigma \right) \ddot{\mathcal{V}}^m - \theta \left( h^{\theta-1} - \frac{2\mathcal{A}^2 \sigma}{p-1} h^{\sigma-1} \right) h' \dot{\mathcal{V}}^m + \frac{\theta}{p-1} h^{-\sigma} ((\nabla^N)^2 V)[\mathcal{V}, E_m] \\ & + \frac{1}{2} \left( h^\theta - \frac{2\mathcal{A}^2 \theta}{p-1} h^\sigma \right) \left( \sum_j (\partial_{jm}^2 g_{11}) \mathcal{V}^j \right) - 2\mathcal{A} \mathcal{A}' \frac{(\theta - \sigma) h^{p-1}}{[(p-1)h^\theta - 2\sigma \mathcal{A}^2 h^\sigma]} H^m \\ & + H^m (\mathbf{H}, \mathcal{V}) \left[ -(p-1) \left( 3 + \frac{\sigma}{\theta} \right) h^{2\theta} - \frac{16\sigma \theta \mathcal{A}^4}{p-1} h^{2\sigma} + 2\mathcal{A}^2 (5\sigma + 3\theta) h^{\theta+\sigma} \right] / [(p-1)h^\theta - 2\mathcal{A}^2 \sigma h^\sigma], \end{aligned} \tag{9}$$

acting on normal sections  $\mathcal{V}$  to  $\gamma$ . Here  $\mathcal{A}'$  stands for the variation of  $\mathcal{A}$  with respect to  $\mathcal{V}$ , and we refer to [7] for the other notations, which are of geometric nature. We can now state our main result:

**Theorem 2.1.** *Let  $M$  be a compact  $n$ -dimensional manifold, let  $V : M \rightarrow \mathbb{R}$  be a smooth positive function and let  $1 < p < \frac{n+1}{n-3}$ . Let  $\gamma$  be a simple closed curve in  $M$ ; then there exists a positive constant  $\mathcal{A}_0$ , depending on  $V|_\gamma$  and  $p$  for which the following holds. If  $0 \leq \mathcal{A} < \mathcal{A}_0$ , if  $\gamma$  satisfies (8) and the operator in (9) is invertible on the normal sections of  $\gamma$ , there is a sequence  $\varepsilon_j \rightarrow 0$  such that problem  $(\text{NLS}_{\varepsilon_j})$  possesses solutions  $\psi_{\varepsilon_j}$  having the asymptotics in (5), with  $f$  satisfying (6).*

Apart from the smallness assumption on  $\mathcal{A}$ , Theorem 2.1 improves the result in [3]. In fact, not only we remove the symmetry condition (which is the main issue, as we shall see), but also the characterization of the limit set is explicit, the assumptions on  $V$  are purely local, and the upper bound on  $p$  is sharp (by a Pohozaev identity for (3)). Our result also extends a previous one in [4] (where real-valued solutions are found for  $n = 2$  only) and show the validity of a conjecture in [2] for 1-dimensional limit sets: in fact as a consequence of Theorem 2.1, one can easily prove the following result:

**Corollary 2.2.** *Let  $(M, g)$ ,  $V$ ,  $p$  be as in Theorem 2.1. Let  $\gamma$  be a simple closed curve which is a nondegenerate geodesic with respect to the weighted metric  $V^{\frac{p+1}{p-1} - \frac{n-1}{2}} g$ . Then there is a sequence  $\varepsilon_j \rightarrow 0$  such that problem  $(\text{NLS}_{\varepsilon_j})$  possesses real-valued solutions  $\psi_{\varepsilon_j}$  concentrating near  $\gamma$  as  $j \rightarrow +\infty$  and having the asymptotic behavior  $\psi_{\varepsilon_j}(\bar{s}, \zeta) \simeq V(\gamma(\bar{s}))^{\frac{1}{p-1}} U(V(\gamma(\bar{s}))^{\frac{1}{2}} \zeta / \varepsilon)$ , where  $\bar{s}$  stands for the arc-length parameter of  $\gamma$ , and  $\zeta$  for a system of normal geodesic coordinates.*

Corollary 2.2 gives also a criterion for the applicability of Theorem 2.1: in fact, starting from a nondegenerate geodesic in the weighted metric, by the implicit function theorem for  $\mathcal{A}$  sufficiently small one obtains a curve for which (8) and the invertibility of (9) hold. In particular, when  $V$  is constant, one can start with nondegenerate close geodesics on  $M$  in the ordinary sense.

### Remarks.

- (i) The statement of Theorem 2.1 remains unchanged if we replace  $M$  by  $\mathbb{R}^n$  (or with an open manifold asymptotically Euclidean at infinity) and we assume  $V$  to be bounded between two positive constants and for which  $\|\nabla^l V\| \leq C_l$ ,  $l = 1, 2, 3$ , for some positive constants  $C_l$ .
- (ii) The existence of solutions to  $(\text{NLS}_{\varepsilon})$  only for a suitable sequence  $\varepsilon_j \rightarrow 0$  is related to a resonance phenomenon, briefly explained below. The result can be extended to a sequence of intervals in the parameter  $\varepsilon$  approaching zero but, at least with our proof, we do not expect to find existence for all  $\varepsilon$ .
- (iii) The restriction on the exponent  $p$  is natural, since (3) admits solitons if and only if  $p$  is subcritical with respect to the dimension  $n - 1$ .
- (iv) The smallness requirement on  $\mathcal{A}$  is technical and we believe this condition could be relaxed. Anyway, for  $\frac{n+2}{n-2} \leq p < \frac{n+1}{n-3}$ ,  $\mathcal{A}$  should have an upper bound depending on  $V$  and  $p$ , to get solvability of both (4) and (6), see Remark 2.3 in [7].
- (v) In a general (nonsymmetric) setting, related results have been achieved in [6] (see also [5]) and [9–11].

### 3. Sketch of the proof of Theorem 2.1

In this section we give the main ideas of the proof of Theorem 2.1, referring to [7] and [8] for full details. The first step in proving our result is to find good approximate solutions using the stationarity and the nondegeneracy conditions on the curve  $\gamma$ . Then, true solutions can be found by a perturbation argument and applying the contraction mapping theorem. After a dilation by a factor  $\frac{1}{\varepsilon}$ , we define  $\tilde{f} = f + \varepsilon f_1$ , we *tilt* the coordinates orthogonal to  $\gamma$  by a normal section  $\Phi$  (setting  $z = \zeta + \Phi$ ) and we look at approximate solution of the form

$$\tilde{\psi}(s, z) = e^{-i \frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \{h(\varepsilon s)U(k(\varepsilon s)z) + \varepsilon[w_r + i w_i] + \varepsilon^2[v_r + i v_i]\}, \quad s \in [0, L(\gamma)/\varepsilon], \quad z \in \mathbb{R}^{n-1}, \quad (10)$$

for some corrections  $w_r, w_i, v_r, v_i$  (here we have set  $\bar{s} = \varepsilon s$ ). To get a formal solvability at order  $\varepsilon$  first and then at order  $\varepsilon^2$ ,  $w_r, w_i, v_r$  and  $v_i$  have to satisfy equations of the form  $\mathcal{L}_r w_r = \mathcal{F}_r$ ,  $\mathcal{L}_i w_i = \mathcal{F}_i$ ,  $\mathcal{L}_r v_r = \tilde{\mathcal{F}}_r$ ,  $\mathcal{L}_i v_i = \tilde{\mathcal{F}}_i$ , where

$$\begin{aligned} \mathcal{L}_r v &= -\Delta_z v + V(\bar{s})v - p h(\bar{s})^{p-1} U(k(\bar{s})z)^{p-1} v; \\ \mathcal{L}_i v &= -\Delta_z v + V(\bar{s})v - h(\bar{s})^{p-1} U(k(\bar{s})z)^{p-1} v, \end{aligned}$$

and where  $\mathcal{F}_r, \mathcal{F}_i, \tilde{\mathcal{F}}_r, \tilde{\mathcal{F}}_i$  are given data which depend on  $V, \gamma, \bar{s}, \mathcal{A}, \Phi$  and  $f_1$ , see Section 3 in [7]. The operators  $\mathcal{L}_r$  and  $\mathcal{L}_i$  are Fredholm (and symmetric) from  $H^2(\mathbb{R}^{n-1})$  into  $L^2(\mathbb{R}^{n-1})$ , and the above equations for the corrections can be solved provided the right-hand sides are orthogonal to the kernels of these operators. It turns out that the kernels of  $\mathcal{L}_r$  and  $\mathcal{L}_i$  produce a sequence of eigenvalues of the linearized operator  $L_\varepsilon$  (at  $\tilde{\psi}$ ) which behave *qualitatively* like  $\varepsilon^2 j^2$  and for small values of  $j$  these become resonant. With an accurate expansion of these eigenvalues, one finds that the nondegeneracy assumption on (9) prevents each of them to vanish: anyway, a direct application of the implicit function theorem is not possible since a further resonance phenomenon occurs. This arises from the fact that  $\mathcal{L}_r$  also possesses a negative eigenvalue  $\eta_0$ , which generates an extra sequence of eigenvalues for  $L_\varepsilon$ , qualitatively of the form  $\eta_0 + \varepsilon^2 j^2$ ,  $j \in \mathbb{N}$ . This resonance is typical of concentration for  $(\text{NLS}_\varepsilon)$  along sets of positive dimension, and the only

hope to get invertibility is to choose the values of  $\varepsilon$  appropriately. Indeed, differently from the previous sequence of eigenvalues, this new one causes a divergence of the Morse index when  $\varepsilon$  tends to zero, and the presence of a kernel for some epsilon's is unavoidable. We then have three possible resonant modes, two of them for small values of the index  $j$ , with eigenvectors  $\partial_l U(k(\bar{s})z)$ ,  $l = 2, \dots, n$ , and  $iU(k(\bar{s})z)$  and a third one for  $j$  of order  $\frac{1}{\varepsilon}$ , precisely when  $\eta_0 + \varepsilon^2 j^2 = 0$ . We notice that, under symmetry assumptions (as in [2] and [3]) resonance can be avoided working with invariant functions.

To deal with the general case, two techniques have been used: the first, employed in [10], is based on a theorem by T. Kato for tracking the possibly resonant eigenvalues, and their dependence on  $\varepsilon$ . The second, used in [4] and [12], is instead based on a more direct approach, by Lyapunov–Schmidt reductions. The latter seems to apply only to the case of one-dimensional limit sets but it turns out to be quite convenient, also for the present case: here we actually combine both the methods.

For  $\delta > 0$  small we introduce a set  $K_\delta$  consisting of approximate eigenfunctions of  $L_\varepsilon$ , see Subsection 2.3 in [8]. We next consider the subspace  $\bar{H}_\varepsilon$  of functions  $\hat{\psi}$  which are orthogonal to  $K_\delta$ , and show that there is invertibility under these restrictions. This indeed allow us to solve (1) up to a Lagrange multiplier in  $K_\delta$ . Our next task will be then to annihilate this Lagrange multiplier: this can be done choosing an approximate solution  $\tilde{\Psi}_\varepsilon$  (more accurate than the one in (10)) which depends on suitable parameters: besides the normal section  $\Phi$ , a phase factor  $f_2$  and a complex-valued function  $\beta$ . The latter parameters correspond to different components of  $K_\delta$ , and are related to the kernels of  $\mathcal{L}_r(+\varepsilon^2 j^2)$  and  $\mathcal{L}_i(+\varepsilon^2 j^2)$ . The function  $\beta$  in particular is highly oscillatory, and takes care of the resonances due to the fast Fourier modes, see [8], Section 4.

In this way, using a Lyapunov–Schmidt reduction onto the set  $K_\delta$  and considering the projections onto its components, we are reduced (roughly) to solve a system of three second-order ordinary differential equations with periodic conditions in  $[0, L(\gamma)]$ . The operators acting on  $\Phi$  and  $f_2$  are rather easy to deal with (to prove invertibility in appropriate functional spaces) using our assumptions on the curve  $\gamma$ , and a natural choice for the functional spaces to work with (for  $\Phi$  and  $f_2$ ) is  $H^2[0, L(\gamma)]$ . Instead, the operator acting on  $\beta$  is more delicate, it is “qualitatively” of the form  $\Lambda_0 \beta := -\varepsilon^2 \beta''(\bar{s}) + \lambda(\bar{s})\beta$  on  $[0, L(\gamma)]$ , with periodic boundary conditions, where  $\lambda$  is a negative function. This operator is precisely the one related to the peculiar resonances described above. In particular it is resonant on frequencies of order  $\frac{1}{\varepsilon}$ , and this requires to choose a norm for  $\beta$  which is weighted in the Fourier modes. For operators like  $\Lambda_0$  there is in general a sequence of epsilon's for which a nontrivial kernel exists. Using Kato's theorem, as in [6,9–11], we can estimate the derivatives of the eigenvalues with respect to  $\varepsilon$ , showing that for several values of this parameter  $\Lambda_0$  is invertible. Once we have this, we apply the contraction mapping theorem to solve the bifurcation equation as well. This yields a solution of our equation (NLS $_\varepsilon$ ).

The first part of the complete paper is devoted to justify assumptions (8), (9) and to find approximate solutions, while the second part contains the proof of the existence result.

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