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Harmonic Analysis

Geometric structure in the representation theory of p-adic groups

Anne-Marie Aubert [a], Paul Baum [b], Roger Plymen [c]

[a] *Institut de Mathématiques de Jussieu, U.M.R. 7586 du C.N.R.S., 175, rue du Chevaleret, 75013 Paris, France*
[b] *Pennsylvania State University, Mathematics Department, University Park, PA 16802, USA*
[c] *School of Mathematics, Manchester University, Manchester M13 9PL, UK*

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Abstract

We conjecture the existence of a simple geometric structure underlying questions of reducibility of parabolically induced representations of reductive p-adic groups. ***To cite this article: A.-M. Aubert et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Structure géométrique en théorie des représentations des groupes p-adiques. Nous conjecturons l'existence d'une structure géométrique simple sous-jacente aux questions de réductibilité des représentations induites paraboliques des groupes réductifs p-adiques. ***Pour citer cet article : A.-M. Aubert et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

Dans cette Note nous conjecturons l'existence d'une structure géométrique simple sous-jacente aux questions de réductibilité des représentations induites paraboliques des groupes réductifs p-adiques.

Considérons un couple (X, Γ), où X est une variété algébrique affine complexe et Γ un groupe fini qui agit sur X comme les automorphismes de la variété algébrique affine X. Nous posons

$$\widetilde{X} := \{(\gamma, x) \in \Gamma \times X \colon \gamma x = x\}.$$

Le groupe Γ agit sur $\widetilde{X}$ par : $\alpha(\gamma, x) = (\alpha\gamma\alpha^{-1}, \alpha x)$, pour $(\gamma, x) \in \widetilde{X}$, $\alpha \in \Gamma$. Rappelons la notion de *quotient étendu* (voir [4]) : le quotient étendu de X par Γ, noté $X/\!/\Gamma$, est défini par $X/\!/\Gamma := \widetilde{X}/\Gamma$, i.e., $X/\!/\Gamma$ est le quotient ordinaire pour l'action de Γ sur $\widetilde{X}$. La projection $\Gamma \times X \to X$, $(\gamma, x) \mapsto x$ définit une application $\pi : X/\!/\Gamma \to X/\Gamma$ surjective à fibres finies qui est un morphisme fini de variétés algébriques.

Soient maintenant F un corps local non archimédien, q le cardinal de son corps résiduel, G le groupe des F-points d'un F-groupe algébrique réductif connexe et $\mathrm{Irr}(G)$ l'ensemble des classes d'équivalence des représentations lisses irréductibles de G. Pour M sous-groupe de Levi de G, nous notons $\mathrm{Cusp}(M)$ l'ensemble des classes d'équivalence des

E-mail addresses: aubert@math.jussieu.fr (A.-M. Aubert), baum@math.psu.edu (P. Baum), plymen@manchester.ac.uk (R. Plymen).

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représentations irréductibles supercuspidales de M et $W(M)$ le groupe $\mathrm{N}_G(M)/M$. Nous appelons *triplet cuspidal* un triplet de la forme (M,σ,w), où M est un sous-groupe de Levi de G, $\sigma\in\mathrm{Cusp}(M)$, $w\in W(M)$, et $w\sigma=\sigma$. Le groupe G agit sur l'ensemble des triplets cuspidaux de la manière suivante : $g\cdot M=gMg^{-1}$, $g\cdot\sigma={}^g\sigma$, $g\cdot w={}^g w$. Soit $\mathfrak{A}(G)$ le quotient par G de l'ensemble des triplets cuspidaux : $\mathfrak{A}(G):=\{(M,\sigma,w)\colon\ w\sigma=\sigma\}/G$. Soit $\Psi(M)$ le groupe des quasicaractères non ramifiés de M et $D:=\Psi(M)\otimes\sigma$.

Pour $w\in W(M)$, nous notons $[\Psi(M)/\mathcal{G}]^w$ l'ensemble des $\psi\in\Psi(M)/\mathcal{G}$ qui sont invariants par w. Cet ensemble a une structure de variété algébrique affine complexe. Puisque $w\cdot(\psi\otimes\sigma)=w\cdot\psi\otimes w\sigma=\psi\otimes w\sigma=\psi\otimes\sigma$, si (M,σ,w) est un triplet cuspidal, il en est de même de $(M,\psi\otimes\sigma,w)$. L'application $[\Psi(M)/\mathcal{G}]^w\to\{(M,\psi\otimes\sigma,w)\colon\ \psi\in[\Psi(M)/\mathcal{G}]^w\}$ est une bijection. Ceci *définit* sur $\mathfrak{A}(G)$ la structure d'une union disjointe d'une famille dénombrable de variétés algébriques affines complexes. Lorsque $G=\mathrm{GL}(n)$, chacune des ces variétés est *lisse*, de dimension d avec $1\leqslant d\leqslant n$. En général, il peut y avoir des variétés singulières.

L'application $(M,\sigma,w)\mapsto(M,\sigma)$ induit une application $\pi:\mathfrak{A}(G)\to\Omega(G)$, de $\mathfrak{A}(G)$ sur la variété de Bernstein $\Omega(G)$ de G formée des classes de G-conjugaison de couples (M,σ), avec M sous-groupe de Levi de G et $\sigma\in\mathrm{Cusp}(M)$.

On a la décomposition de Bernstein $\mathfrak{A}(G)=\bigsqcup\mathfrak{A}(G)^{\mathfrak{s}}$, l'union disjointe étant prise sur les composantes $\mathfrak{s}$ de $\Omega(G)$. La restriction à $\mathfrak{A}(G)^{\mathfrak{s}}$ de l'application $\mathfrak{A}(G)\to\Omega(G)$ est donnée par la projection standard $\mathfrak{A}(G)^{\mathfrak{s}}=D^{\mathfrak{s}}/\!/W^{\mathfrak{s}}\to D^{\mathfrak{s}}/W^{\mathfrak{s}}$.

Pour $\mathfrak{s}$ une composante fixée de $\Omega(G)$, nous posons $D=D^{\mathfrak{s}}$ et $W=W^{\mathfrak{s}}$. Nous munissons la variété quotient D/W de la topologie de Zariski et $\mathrm{Irr}(G)^{\mathfrak{s}}$ (la $\mathfrak{s}$-composante de $\mathrm{Irr}(G)$ dans sa décomposition de Bernstein) de celle de Jacobson. Remarquons que le fait d'être irréductible étant une condition *ouverte* l'ensemble $\mathfrak{R}$ des $(M,\psi\otimes\sigma)$ tels que l'induite parabolique de $\psi\otimes\sigma$ est réductible est une sous-variété de D/W. Notons E le sous-groupe compact maximal de D, inf.ch. le caractère infinitésimal [5]. Notons $\mathfrak{X}_{\mathrm{red}}$ le schéma réduit associé à $\mathfrak{X}$. Dans le contexte présent, nous appellerons *co-caractère* un homomorphisme de groupes algébriques $\mathbb{C}^\times\to\Psi(M)$.

Conjecture 0.1.

(1) *Il existe une famille plate $\mathfrak{X}_t$ de sous-schémas de D/W, avec $t\in\mathbb{C}^\times$, telle que $\mathfrak{X}_1=\pi(D/\!/W-D/W)$ et $\mathfrak{X}_{\sqrt{q}}=\mathfrak{R}$.*
(2) *Pour toute composante irréductible $\mathbf{c}$ de $D/\!/W$, il existe un co-caractère $h_{\mathbf{c}}:\mathbb{C}^\times\to\Psi(M)$ tel que, si nous posons $\pi_t(x)=\pi(h_{\mathbf{c}}(t)\otimes x)$ pour $x\in\mathbf{c}$, alors, pour tout $t\in\mathbb{C}^\times$, $\pi_t:D/\!/W\to D/W$ est un morphisme fini satisfaisant $(\mathfrak{X}_t)_{\mathrm{red}}=\pi_t(D/\!/W-D/W)$. Si $\mathbf{c}=D/W$ alors $h_{\mathbf{c}}=1$. Les schémas $\mathfrak{X}_1,\mathfrak{X}_{\sqrt{q}}$ sont réduits.*
(3) *Il existe une bijection continue $\mu:D/\!/W\to\mathrm{Irr}(G)^{\mathfrak{s}}$ telle que* (inf.ch.) $\circ\,\mu=\pi_{\sqrt{q}}$ *et* $\mu(E/\!/W)=\mathrm{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}}$.

Théorème 0.1. *La conjecture est vraie pour $G=\mathrm{SL}(2)$ et pour $G=\mathrm{GL}(n)$.*

Nous avons d'autre part choisi d'illustrer notre conjecture par le cas des représentations du groupe exceptionnel de type G_2 qui possèdent des vecteurs non nuls invariants par un sous-groupe d'Iwahori :

Théorème 0.2. *La conjecture est vraie pour le point $\mathfrak{s}=[T,1]_{G_2}$.*

1. Introduction

In the representation theory of reductive p-adic groups, the issue of reducibility of induced representations is an issue of great intricacy: see, for example, the classic article by Bernstein and Zelevinsky [6] on $\mathrm{GL}(n)$ and the more recent article by Muić [10] on G_2. It is our contention, expressed as a conjecture, that there exists a simple geometric structure underlying this intricate theory.

For the moment, our conjecture is *local*, in that it applies only to finite places. To explain our conjecture, we need to refine the usual concept of quotient space.

2. The extended quotient

We will recall the concept of *extended quotient* [4]. Let Γ be a finite group and let X be a complex affine algebraic variety. Assume that Γ is acting on X as automorphisms of the affine algebraic variety X. Let

$$\widetilde{X} := \{(\gamma, x) \in \Gamma \times X \colon \gamma x = x\}.$$

The group Γ acts on $\widetilde{X}$ by:

$$\alpha(\gamma, x) = (\alpha\gamma\alpha^{-1}, \alpha x) \quad \text{with } (\gamma, x) \in \widetilde{X},\ \alpha \in \Gamma.$$

Definition 2.1. The extended quotient, denoted $X//\Gamma$, is defined as

$$X//\Gamma := \widetilde{X}/\Gamma,$$

i.e. $X//\Gamma$ is the ordinary quotient for the action of Γ on $\widetilde{X}$.

The projection $\Gamma \times X \to X$, $\quad (\gamma, x) \mapsto x$ gives a map $\pi : X//\Gamma \to X/\Gamma$ called *the projection of the extended quotient on the ordinary quotient*. This is a finite morphism of algebraic varieties.

Let $e \in \Gamma$ be the neutral element. The map $x \mapsto (x, e)$ induces injective morphisms $X \to \widetilde{X}$ and $X/\Gamma \to X//\Gamma$. We shall view X/Γ as a sub-variety of $X//\Gamma$. The complement of X/Γ in $X//\Gamma$ will be denoted $X//\Gamma - X/\Gamma$.

3. The extended variety $\mathfrak{A}(G)$

Let F be a local nonarchimedean field, let G be the group of F-rational points in a connected reductive algebraic group defined over F, and let $\mathrm{Irr}(G)$ be the set of equivalence classes of irreducible smooth representations of G. For M a Levi subgroup of G, we denote by $\mathrm{Cusp}(M)$ the set of equivalence classes of irreducible supercuspidal representations of M and by $W(M)$ the group $\mathrm{N}_G(M)/M$. By a *cuspidal triple* we shall mean a triple of the form (M, σ, w), where M is a Levi subgroup of G, $\sigma \in \mathrm{Cusp}(M)$, $w \in W(M)$, and $w\sigma = \sigma$. The group G acts on the set of all cuspidal triples:

$$g \cdot M = gMg^{-1}, \quad g \cdot \sigma = {}^g\sigma, \quad g \cdot w = {}^g w.$$

Denote by $\mathfrak{A}(G)$ the quotient by G of the set of all cuspidal triples:

$$\mathfrak{A}(G) := \{(M, \sigma, w) \colon w\sigma = \sigma\}/G.$$

We recall standard notation of Bernstein [5]: $\Psi(M)$ is the group of unramified quasicharacters of M,

$$D := \Psi(M) \otimes \sigma \subset \mathrm{Irr}(M).$$

We recall that $\Psi(M)$ has the structure of complex torus. There is a short exact sequence (depending on the base point σ) $1 \to \mathcal{G} \to \Psi(M) \to D \to 1$ where $\mathcal{G}$ is a finite subgroup of $\Psi(M)$.

Denote by $[\Psi(M)/\mathcal{G}]^w$ the w-fixed set, $w \in W(M)$. This has the structure of complex affine algebraic variety. Now hold M and w fixed, and consider $\{(M, \psi \otimes \sigma, w) \colon \psi \in [\Psi(M)/\mathcal{G}]^w\}$. Note that

$$w \cdot (\psi \otimes \sigma) = w \cdot \psi \otimes w\sigma = \psi \otimes w\sigma = \psi \otimes \sigma$$

so that the new triples are cuspidal triples. The map $[\Psi(M)/\mathcal{G}]^w \to \{(M, \psi \otimes \sigma, w) \colon \psi \in [\Psi(M)/\mathcal{G}]^w\}$ is a bijection. This *defines* on $\mathfrak{A}(G)$ the structure of a disjoint union of countably many complex affine algebraic varieties. When $G = \mathrm{GL}(n)$, each of these varieties is *smooth*, of dimension d with $1 \leqslant d \leqslant n$. In general, the varieties may be singular.

We have a map from $\mathfrak{A}(G)$ to the Bernstein variety $\Omega(G)$, induced by the map $(M, \sigma, w) \mapsto (M, \sigma)$ which sends a cuspidal triple to the corresponding cuspidal pair. We denote this by

$$\pi : \mathfrak{A}(G) \to \Omega(G).$$

This determines the Bernstein decomposition of $\mathfrak{A}(G)$:

$$\mathfrak{A}(G) = \bigsqcup \mathfrak{A}(G)^{\mathfrak{s}}$$

the disjoint union taken over all the components $\mathfrak{s}$ of $\Omega(G)$. The map $\mathfrak{A}(G) \to \Omega(G)$, restricted to $\mathfrak{A}(G)^{\mathfrak{s}}$ is given by the standard projection

$$\mathfrak{A}(G)^{\mathfrak{s}} = D^{\mathfrak{s}} /\!/ W^{\mathfrak{s}} \to D^{\mathfrak{s}} / W^{\mathfrak{s}}.$$

We will fix a component $\mathfrak{s}$ of $\Omega(G)$ and write $D = D^{\mathfrak{s}}$, $W = W^{\mathfrak{s}}$. Let $\mathrm{Irr}(G)^{\mathfrak{s}}$ denote the $\mathfrak{s}$-component of $\mathrm{Irr}(G)$ in the Bernstein decomposition of $\mathrm{Irr}(G)$. We will give the quotient variety D/W the Zariski topology, and $\mathrm{Irr}(G)^{\mathfrak{s}}$ the Jacobson topology. We note that irreducibility is an *open* condition, and so the set $\mathfrak{R}$ of reducible points in D/W, i.e. those $(M, \psi \otimes \sigma)$ such that when parabolically induced to G, $\psi \otimes \sigma$ becomes reducible, is a sub-variety of D/W. Let q denote the cardinality of the residue field of F. Let E be the maximal compact subgroup of D, let *inf.ch.* be the infinitesimal character of Bernstein [5]. The reduced scheme associated to a scheme $\mathfrak{X}$ will be denoted $\mathfrak{X}_{\mathrm{red}}$ as in [8, p. 25]. In the present context, a *cocharacter* will mean a homomorphism of algebraic groups $\mathbb{C}^\times \to \Psi(M)$.

Conjecture 3.1.

(1) *There is a flat family $\mathfrak{X}_t$ of subschemes of D/W, with $t \in \mathbb{C}^\times$, such that*

$$\mathfrak{X}_1 = \pi(D/\!/W - D/W), \qquad \mathfrak{X}_{\sqrt{q}} = \mathfrak{R}.$$

(2) *For each irreducible component $\mathbf{c} \subset D/\!/W$ there is a cocharacter $h_{\mathbf{c}}: \mathbb{C}^\times \to \Psi(M)$ such that, if we set $\pi_t(x) = \pi(h_{\mathbf{c}}(t) \otimes x)$ for all $x \in \mathbf{c}$, then, for each $t \in \mathbb{C}^\times$, $\pi_t : D/\!/W \to D/W$ is a finite morphism with $(\mathfrak{X}_t)_{\mathrm{red}} = \pi_t(D/\!/W - D/W)$. If $\mathbf{c} = D/W$ then $h_{\mathbf{c}} = 1$. The schemes $\mathfrak{X}_1, \mathfrak{X}_{\sqrt{q}}$ are reduced.*
(3) *There exists a continuous bijection $\mu: D/\!/W \to \mathrm{Irr}(G)^{\mathfrak{s}}$ with $(inf.ch.) \circ \mu = \pi_{\sqrt{q}}$ and with $\mu(E/\!/W) = \mathrm{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}}$.*

Theorem 3.1. *The conjecture is true for $G = \mathrm{SL}(2)$. If $\mathfrak{s} = [T, 1]_G$ then $\mathfrak{X}_t$ is the 0-dimensional variety given by the Laurent polynomial $(x+1)(x^{-1}+1)(x-t^2)(x^{-1}-t^2) = 0$. When t is the fourth root of unity i or $-\mathrm{i}$, this scheme is the double point given by $(x+1)^2(x^{-1}+1)^2 = 0$.*

4. The general linear group

Theorem 4.1. *The conjecture is true for $G = \mathrm{GL}(n)$.*

Proof. The proof uses Langlands parameters, together with some careful combinatorics. In effect, the L-parameters encode the extended quotient for $\mathrm{GL}(n)$. The details of the proof appear in [2] and [7].

Let $G = \mathrm{GL}(n) = \mathrm{GL}(n, F)$, $n = mr$, $\tau \in \mathrm{Cusp}(\mathrm{GL}(m, F))$, $\mathfrak{s} = [M, \sigma]_G = [\mathrm{GL}(m)^r, \tau^{\otimes r}]_G$. We have $D = D^{\mathfrak{s}} = (\mathbb{C}^\times)^r$, $W = W^{\mathfrak{s}} = S_r$. Let W_F be the Weil group of F, and let $\mathcal{L}_F = W_F \times \mathrm{SU}(2)$. Let $\Phi(G)$ denote the set of equivalence classes of Frobenius-semisimple smooth homomorphisms from $\mathcal{L}_F$ to $\mathrm{GL}(n, \mathbb{C})$. For each $n \geqslant 1$ we have the local Langlands correspondence [9]

$$\mathrm{rec}_F : \mathrm{Irr}\big(\mathrm{GL}(n, F)\big) \to \Phi(G).$$

Now let $\mathrm{rec}_F(\tau) = \eta \in \mathrm{Irr}_m(W_F)$. Denote by $R(j)$ the j-dimensional irreducible complex representation of $\mathrm{SU}(2)$. Let $w \in S_r$ be a product of cycles of different lengths $a_1, \ldots, a_l$, with a_j repeated r_j times. Corresponding to w we have the L-parameter

$$\phi := \eta \otimes R(a_1) \oplus \cdots \oplus \eta \otimes R(a_1) \oplus \cdots \oplus \eta \otimes R(a_l) \oplus \cdots \oplus \eta \otimes R(a_l) \tag{1}$$

where $\eta \otimes R(a_j)$ is repeated r_j times. We will now give each direct summand in the above expression an unramified twist, by *unramified quasicharacters* ψ of W_F. We will map the resulting L-parameters as follows:

$$\psi_1 \otimes \eta \otimes R(a_1) \oplus \cdots \oplus \psi_{r_1+\cdots+r_l} \otimes \eta \otimes R(a_l) \mapsto \big(\psi_1(\varpi_F), \ldots, \psi_{r_1+\cdots+r_l}(\varpi_F)\big) \in D^\gamma$$

where ϖ_F is a uniformizer in F. Let $\Phi(G)^{\mathfrak{s}}$ denote the $\mathfrak{s}$-component of $\Phi(G)$ in the Bernstein decomposition of $\Phi(G)$, so that $\Phi(G)^{\mathfrak{s}} = \mathrm{rec}_F(\mathrm{Irr}(G)^{\mathfrak{s}})$.

We now take the disjoint union of the permutations w, one chosen in each W-conjugacy class. This creates a *canonical* bijection

$$\alpha : \Phi(G)^{\mathfrak{s}} \cong D//W.$$

Our map μ is then defined as follows:

$$\mu = \mathrm{rec}_F^{-1} \circ \alpha^{-1} : D//W \to \mathrm{Irr}(G)^{\mathfrak{s}}.$$

The sub-variety $\pi(D//W - D/W)$ is the hypersurface $\mathfrak{X}_1$ given by the single equation $\prod_{i \neq j}(z_i - z_j) = 0$. The variety $\mathfrak{R}$ is the variety $\mathfrak{X}_{\sqrt{q}}$ given by the single equation $\prod_{i \neq j}(z_i - qz_j) = 0$, according to a classical theorem [6, Theorem 4.2], [13]. The polynomial equation $\prod_{i \neq j}(z_i - t^2 z_j) = 0$ determines a flat family $\mathfrak{X}_t$ of hypersurfaces. The hypersurface $\mathfrak{X}_1$ is the *flat limit* of the family $\mathfrak{X}_t$ as $t \to 1$, as in [8, p. 77]. Let $\mathbf{c}$ be the G-orbit of the cuspidal triple $(\mathrm{GL}(m)^r, \tau^{\otimes r}, w)$, so that $\mathbf{c}$ is an irreducible component in $\mathfrak{A}(\mathrm{GL}(n))$. Note that the L-parameter ϕ in Eq. (1) can be written $\phi = \eta \otimes g$ with g an r-dimensional representation of $\mathrm{SL}(2, \mathbb{C})$. The cocharacter $h_{\mathbf{c}}$ is given by restriction of g to the diagonal subgroup:

$$t \mapsto g\bigl(\mathrm{diag}(t, t^{-1})\bigr) \in (\mathbb{C}^\times)^r$$

and we infer that $(\mathit{inf.ch.}) \circ \mu = \pi_{\sqrt{q}}$. □

Let $\mu^G(\omega)\mathrm{d}\omega$ denote Plancherel measure, with the canonical measure $\mathrm{d}\omega$ normalized as in [12]. According to the explicit Plancherel formula in [1], itself based on formulas of Harish-Chandra and Langlands–Shahidi, the Plancherel density $\mu^{\mathrm{GL}(n)}$ extends uniquely to a *rational* function on the extended variety $\mathfrak{A}(\mathrm{GL}(n))$. In this sense, the extended variety $\mathfrak{A}(\mathrm{GL}(n))$ is a natural domain of $\mu^{\mathrm{GL}(n)}$.

5. The Iwahori spherical representations of G_2

We have chosen the exceptional group G_2 as an example, requiring many delicate calculations, see [3]. We will need a detailed analysis of the Iwahori spherical representations [10,11]. Let $\mathfrak{s} = [T, 1]_G$ where $T \simeq F^\times \times F^\times$ is a maximal F-split torus of $G = G_2$. We note that $\Psi(T) \cong T^\vee$ with $T^\vee$ a maximal torus in the Langlands dual group $G^\vee = G_2(\mathbb{C})$. The Weyl group W of G_2 is the dihedral group of order 12. The extended quotient is

$$T^\vee//W = T^\vee/W \sqcup \mathfrak{C}_1 \sqcup \mathfrak{C}_2 \sqcup \mathrm{pt}_1 \sqcup \mathrm{pt}_2 \sqcup \mathrm{pt}_3 \sqcup \mathrm{pt}_4 \sqcup \mathrm{pt}_5.$$

The flat family is $\mathfrak{X}_t := (1 - t^2 y)(x - t^2 y) = 0$. Note that $\mathfrak{X}_{\sqrt{q}} = \mathfrak{R}$ the curve of reducibility points in the quotient variety $T^\vee/W$. The restriction of π_t to $T^\vee//W - T^\vee/W$ determines a finite morphism

$$\mathfrak{C}_1 \sqcup \mathfrak{C}_2 \sqcup \mathrm{pt}_1 \sqcup \mathrm{pt}_2 \sqcup \mathrm{pt}_3 \sqcup \mathrm{pt}_4 \sqcup \mathrm{pt}_5 \longrightarrow \mathfrak{X}_t.$$

Example. The fibre of the point $(q^{-1}, 1) \in \mathfrak{R}$ via the map $\pi_{\sqrt{q}}$ is a set with 5 points, corresponding to the fact that there are 5 smooth irreducible representations of G_2 with infinitesimal character $(q^{-1}, 1)$.

The map π_t restricted to the one affine line $\mathfrak{C}_1$ is induced by the map $(z, 1) \mapsto (tz, t^{-2})$, and restricted to the other affine line $\mathfrak{C}_2$ is induced by the map $(z, z) \mapsto (tz, t^{-1}z)$. With regard to the second map: the two points $(\omega/\sqrt{q}, \omega/\sqrt{q})$, $(\omega^2/\sqrt{q}, \omega^2/\sqrt{q})$ are distinct points in $\mathfrak{C}_2$ but become identified via $\pi_{\sqrt{q}}$ in the quotient variety $T^\vee/W$. This implies that the image $\pi_{\sqrt{q}}(\mathfrak{C}_2)$ of one affine line has a *self-intersection point* in the quotient variety $T^\vee/W$. Also, the curves $\pi_{\sqrt{q}}(\mathfrak{C}_1), \pi_{\sqrt{q}}(\mathfrak{C}_2)$ intersect in 3 points. These intersection points account for the number of distinct constituents in the corresponding induced representations.

Theorem 5.1. *The conjecture is true for the point* $\mathfrak{s} = [T, 1]_{G_2}$.

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