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C. R. Acad. Sci. Paris, Ser. I 345 (2007) 495–497

COMPTES RENDUS MATHEMATIQUE

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Algebraic Geometry

# An analog of a theorem of Lange and Stuhler for principal bundles

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Received 12 July 2007; accepted after revision 2 October 2007

Available online 31 October 2007

Presented by Michel Raynaud

**Abstract**

Let $k$ be an algebraically closed field of characteristic $p > 0$ and $G$ the base change to $k$ of a connected reduced linear algebraic group defined over $\mathbb{Z}/p\mathbb{Z}$. Let $E_G$ be a principal $G$-bundle over a projective variety $X$ defined over the field $k$. Assume that there is an étale Galois covering $f : Y \to X$ with degree$(f)$ coprime to $p$ such that the pulled back principal $G$-bundle $f^*E_G$ is trivializable. Then there is a positive integer $n$ such that the pullback $(F_X^n)^*E_G$ is isomorphic to $E_G$, where $F_X$ is the absolute Frobenius morphism of $X$.

This can be considered as a weak converse of the following observation due to P. Deligne. Let $H$ be any algebraic group defined over $k$ and $E_H$ a principal $H$-bundle over a scheme $Z$. If the pulled back principal $H$-bundle $(F_Z^n)^*E_H$ over $Z$ is isomorphic to $E_H$ for some $n > 0$, where $F_Z$ is the absolute Frobenius morphism of $Z$, then there is a finite étale Galois cover of $Z$ such that the pullback of $E_H$ to it is trivializable. ***To cite this article: I. Biswas, L. Ducrohet, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***


**Résumé**

**Un analogue d'un théorème de Lange et Stuhler pour les fibrés principaux.** Soient $k$ un corps algébriquement clos de caractéristique positive $p$ et $G$ l'extension à $k$ d'un groupe linéaire algébrique connexe, défini sur $\mathbb{Z}/p\mathbb{Z}$. Soit $E_G$ un $G$-fibré principal au-dessus d'une variété projective $X$ défini sur $k$. Supposons qu'il existe un revêtement étale galoisien $f : Y \to X$ de degré premier à $p$ tel que le pull-back $f^*E_G$ est trivial. Alors il existe un entier $n > 0$ tel que le pull-back $(F_X^n)^*E_G$ est isomorphe à $E_G$, où $F_X$ est le Frobenius absolu de $X$.

Ce résultat peut être considéré comme une réciproque partielle de l'observation suivante due à P. Deligne. Soit $H$ un groupe algébrique quelconque défini sur $k$ et $E_H$ un $H$-fibré principal au-dessus d'un schéma $Z$. Si l'image inverse $(F_Z^n)^*E_H$ est isomorphe à $E_H$ pour un entier $n > 0$ convenable, alors il existe un revêtement étale galoisien de $Z$ tel que le pull-back de $E_H$ à ce revêtement est trivial, où $F_Z$ est le Frobenius absolu de $Z$. ***Pour citer cet article : I. Biswas, L. Ducrohet, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***


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1631-073X/$ – see front matter 
doi:10.1016/j.crma.2007.10.010

## 1. Introduction

Let $X$ be a projective variety defined over an algebraically closed field $k$ of positive characteristic. Let $F_X : X \to X$ be the absolute Frobenius morphism of $X$. For any integer $n \geqslant 1$, the $n$-fold iteration of the self-map $F_X$ will be denoted by $F_X^n$.

A vector bundle, or more generally a principal bundle, over $X$ is called étale trivializable if its pullback to some étale Galois cover of $X$ is a trivial bundle. The following theorem is due to Lange and Stuhler (see [3]):

**Theorem 1.1.** *If a vector bundle $E$ over $X$ is isomorphic to $(F_X^n)^*E$ for some $n \geqslant 1$, then $E$ is étale trivializable. For any stable étale trivializable vector bundle $E$ over $X$, there is a positive integer $n$ such that $(F_X^n)^*E$ is isomorphic to $E$.*

See [3, p. 75, Theorem. 1.4] for the proof of the first part. For the second part, assume that $E$ is trivialized by an étale Galois covering $Y \to X$ with Galois group $\Gamma$ of order $d$. Then there is a group homomorphism

$$\rho : \Gamma \to \mathrm{GL}\big(\mathrm{rk}(E), k\big)$$

giving $E$; see also (1). Since $\Gamma$ has order $d$, the determinant of any element in the image of $\rho$ is in the finite field $\mathbb{F}_{p^d}$, and the determinant is a character of $\mathrm{GL}(\mathrm{rk}(E), k)$. Also, the stability condition of $E$ implies that $\rho$ is absolutely irreducible. Therefore, from [1, p. 150, Theorem 9.14] it follows that $\rho$ is actually equivalent to a representation $G \to \mathrm{GL}(\mathrm{rk}(E), \mathbb{F}_{p^d})$. This implies that $(F_X^d)^*E \cong E$.

Although the condition in the second statement of Theorem 1.1 that $E$ is stable was not mentioned in [3], the following example provided by Y. Laszlo shows that the second statement of Theorem 1.1 is not valid if the assumption that $E$ is stable is removed.

Take a Mumford–Tate curve $X$ of genus $g \geqslant 3$ over $\overline{\mathbb{F}_p((t))}$. Its fundamental group is isomorphic to the profinite completion of the free group with $g$ generators and it surjects onto the finite group $G$ of all upper triangular unipotent $3 \times 3$ matrices with entries in $\mathbb{F}_p$. Let $\pi : Y \to X$ be an étale Galois covering with Galois group $G$. One can check that the representation of $G$ in $\mathrm{GL}(3, \overline{\mathbb{F}_p((t))})$ defined by

$$\rho : \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & at & bt^2 \\ 0 & 1 & ct \\ 0 & 0 & 1 \end{pmatrix}$$

is not conjugated to any of its Frobenius twists. Therefore, the rank 3 vector bundle $(\pi_*(\mathcal{O}_Y{}^{\oplus 3}))^{\rho(G)}$ on $X$ associated to this representation is étale trivializable but it is not isomorphic to any of its iterated Frobenius pullbacks.

Let $G_p$ be any connected reduced linear algebraic group defined over $\mathbb{Z}/p\mathbb{Z}$, where $p$ is the characteristic of $k$. Let $G := G_p \times_{\mathbb{F}_p} k$ be the base change of $G_p$ to $k$. Let $E_G$ be a principal $G$-bundle over $X$.

Our aim here is to prove the following:

**Theorem 1.2.** *Assume that there is an étale Galois covering $f : Y \to X$ such that*

- *$f^*E_G$ is trivializable, and*
- degree$(f)$ *is coprime to the characteristic of the field $k$.*

*Then there is a positive integer $n$ such that the pullback $(F_X^n)^*E_G$ is isomorphic to $E_G$.*

This theorem is proved in Section 2.

Let $H$ be any algebraic group defined over $k$. Let $E_H$ be a principal $H$-bundle over a scheme $Z$. If the pulled back principal $H$-bundle $(F_Z^n)^*E_H$ over $Z$ is isomorphic to $E_H$ for some $n > 0$, where $F_Z$ is the absolute Frobenius morphism of $Z$, then $E_H$ is étale trivializable [4, p. 655] (in [4] this is attributed to P. Deligne).

## 2. Étale trivializable with degree coprime to characteristic

Let $G$ be the base change to $k$ of a connected reduced linear algebraic group $G_p$ defined over the field $\mathbb{F}_p$, where $p$ is the characteristic of $k$. As in the previous section, $E_G$ is a principal $G$-bundle over the projective variety $X$.

**Theorem 2.1.** *Assume that there is an étale Galois covering $f: Y \to X$ such that the pulled back principal $G$-bundle $f^*E_G$ is trivializable, and the degree of $f$ is coprime to $p$. Then there is a positive integer $n$ such that the pullback $(F_X^n)^*E_G$ is isomorphic to $E_G$.*

**Proof.** Let $\Gamma := \mathrm{Gal}(f)$ be the Galois group for $f$. Fixing a trivialization

$$\psi : f^*E_G \to Y \times G$$

of $f^*E_G$, and also choosing a $k$-rational point $y_0$ of $Y$, we get a homomorphism

$$\rho : \Gamma \to G \tag{1}$$

which is constructed as follows. For any $\gamma \in \Gamma$, consider the natural identifications of the fiber $(E_G)_{f(y_0)}$ with $(f^*E_G)_{y_0}$ and $(f^*E_G)_{\gamma(y_0)}$; in terms of these identifications,

$$\psi^{-1}\big((\gamma(y_0), e)\big) = \psi^{-1}\big((y_0, e)\big)\rho(\gamma).$$

It is easy to see that $E_G$ is identified with the extension of structure group, using $\rho$, of the principal $\Gamma$-bundle $Y$ over $X$.

Let $\mathfrak{g}$ denote the Lie algebra of $G$. The adjoint action of $G$ on $\mathfrak{g}$ and the homomorphism $\rho$ in (1) together define an action of $\Gamma$ on $\mathfrak{g}$. Since the cardinality $\#\Gamma$ is coprime to the characteristic of the field $k$, we have

$$H^1(\Gamma, \mathfrak{g}) = 0 \tag{2}$$

(see [2, p. 83, Exercise 2]).

Let

$$M_h := \mathrm{Hom}(\Gamma, G)$$

be the variety comprising of all homomorphisms from $\Gamma$ to $G$. It is known that $M_h$ is an affine variety. Also, $M_h$ is the base change to $k$ of the variety, defined over $\mathbb{F}_p$, given by the homomorphisms from $\Gamma$ to $G_p$. The group $G$ acts on $M_h$ through conjugation. Let

$$O(\rho) \subset M_h$$

be the orbit of the homomorphism $\rho$ in (1) for this action of $G$ on $M_h$. Using (2) it follows that $O(\rho)$ is a Zariski open subset [6]. (For any morphism from a smooth variety to a scheme such that at some point the homomorphism of Zariski tangent spaces is surjective, the image contains a Zariski open subset; using this together with [5, p. 91, Lemma 6.8] it follows that $O(\rho)$ is a Zariski open subset of $M_h$.) Therefore, there is some positive integer $n$ such that $O(\rho)$ contains a homomorphism

$$\rho_n : \Gamma \to G_{p^n}, \tag{3}$$

where $G_{p^n}$ is the base change of $G_p$ to $\mathbb{F}_{p^n}$. In other words, the point $\rho_n$ of $M_h$ is defined over $\mathbb{F}_{p^n}$.

Let $E_G^n$ denote the principal $G$-bundle over $X$ obtained by extending the structure group of the principal $\Gamma$-bundle $Y$ using the homomorphism $\rho_n$ in (3). Since $\rho_n$ is conjugate to $\rho$, it follows that $E_G^n$ is isomorphic to $E_G$. On the other hand, the pulled back principal $G$-bundle $(F_X^n)^*E_G^n$ is isomorphic to $E_G^n$ because the homomorphism $\rho_n$ is defined over $\mathbb{F}_{p^n}$. This completes the proof of the theorem. □


## Acknowledgements

We thank Y. Laszlo for the example in Section 1.


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