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Mathematical Analysis

The Schur–Szegö composition for hyperbolic polynomials ☆

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Abstract

The composition of Schur–Szegö of the polynomials $P(x)=\sum_{j=0}^{n}C_n^ja_jx^j$ and $Q(x)=\sum_{j=0}^{n}C_n^jb_jx^j$ is defined as $P*Q=\sum_{j=0}^{n}C_n^ja_jb_jx^j$. In the case when P and Q are hyperbolic, i.e. with real roots only, we give the exhaustive answer to the question if the numbers of positive, negative and zero roots of P and Q are known what these numbers can be for $P*Q$. ***To cite this article: V.P. Kostov, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

La composition de Schur–Szegö de polynômes hyperboliques. La composition de Schur–Szegö des polynômes $P(x)=\sum_{j=0}^{n}C_n^ja_jx^j$ et $Q(x)=\sum_{j=0}^{n}C_n^jb_jx^j$ est définie comme $P*Q=\sum_{j=0}^{n}C_n^ja_jb_jx^j$. Dans le cas où P et Q sont hyperboliques, c. à d. n'ayant que des racines réelles, nous donnons la réponse exhaustive à la question si on connaît les nombres de racines positives, négatives et nulles de P et Q, quels peuvent être ces nombres pour $P*Q$. ***Pour citer cet article : V.P. Kostov, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

La *composition de Schur–Szegö* des polynômes $P(x)=\sum_{j=0}^{n}C_n^ja_jx^j$ et $Q(x)=\sum_{j=0}^{n}C_n^jb_jx^j$ est définie par la formule $P\,{}^*_n\,Q=\sum_{j=0}^{n}C_n^ja_jb_jx^j$. Dans cet article on a $k\in\mathbf{N}\cup 0$.

Remarque 1. Si on considère P et Q comme des polynômes de degré $n+1$ à coefficients dominants 0, leur composition devrait être définie par une formule différente : $P\,{}^*_{n+1}\,Q=\sum_{j=0}^{n+1}((C_n^j)^2/C_{n+1}^j)a_jb_jx^j$. L'indice n sous $*$ est mis pour éviter cette ambiguité. On peut interpréter $R_{n,k}:=P\,{}^*_{n+k}\,Q$ comme la composition de deux polynômes ayant chacun une racine de multiplicité k à l'infini.

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Nous considérons le cas où P et Q sont *hyperboliques*, c'est-à-dire dont toutes les racines sont réelles. Dans ce cas nous donnons la réponse exhaustive à la question si on connaît le nombre de racines positives, négatives et nulles de P et de Q, quels peuvent être ces nombres pour le polynôme $R_{n,k}$, voir Théorèmes 0.2 et 0.3. On désigne par $H_{u,v,w}$ l'ensemble des polynômes hyperboliques de degré n ayant respectivement u, v et w racines strictement négatives, nulles et strictement positives.

Proposition 0.1. (1) *Si $R_{n,0} \in H_{u,v,w}$, alors pour tout k on a $R_{n,k} \in H_{u,v,w}$ et $H_{u,v,w} {}_{n+k}^{*} H_{n,0,0} \subset H_{u,v,w}$. Si $R_{n,0}$ n'est pas forcément hyperbolique et a u' racines < 0, w' racines < 0 et une racine de multiplicité v' en 0, alors pour tout k, $R_{n,k}$ a au moins u' racines strictement négatives, au moins w' racines strictement positives et une racine de multiplicité v' en 0.*

(2) *Si $x_P \neq 0$ et $x_Q \neq 0$ sont racines de P et Q de multiplicité m_P et m_Q, $m_P + m_Q \geqslant n + k$, alors $-x_P x_Q$ est racine de $R_{n,k}$ de multiplicité $m_P + m_Q - n - k$.*

Théorème 0.2. *Supposons que P et Q sont deux polynômes complexes de degré n tels que $Q = x^q S$, $\deg(S) = n - q$. Alors pour tout k on a $P {}_{n+k}^{*} Q = \frac{(n+k-q)!}{(n+k)!} x^q (P^{(q)} {}_{n+k-q}^{*} S)$.*

Remarque 2. Dans le cas $k = 0$, $P \in H_{u,v,w}$, $Q \in H_{n,0,0}$ ou $Q \in H_{0,0,n}$, le vecteur multiplicité (VM) de $R_{n,0}$ (c'est-à-dire le vecteur dont les composantes sont les multiplicités des racines distinctes d'un polynôme hyperbolique données dans l'ordre de croissance) ne dépend que des VM de P et Q, voir [2], Proposition 1.4 et Théorème 1.6. Dans les conditions du théorème, $R_{n,0}$ est hyperbolique et son VM dépend des VM de $P^{(q)}$ et Q.

Remarque 3. On a $P(\alpha x) {}_{n+k}^{*} Q(\beta x) = R_{n,k}(\alpha\beta x)$ (pour tout k et pour tout $\alpha, \beta \in \mathbf{R}^*$).

Considérons le cas $P \in H_{g,0,l}$, $Q \in H_{r,0,s}$. On peut assumer que $g \geqslant l$ et $s = \min(g, l, r, s)$ (si nécessaire on peut appliquer Remarque 3 avec $\alpha = -1$ et/ou $\beta = -1$).

Théorème 0.3.

(1) *Dans ce cas pour tout k, $R_{n,k}$ a au moins $g - s$ racines < 0 et au moins $l - s$ racines > 0 (comptées avec multiplicité) et pas plus de s couples conjugués.*
(2) *Pour tout k et pour tous $\lambda, \nu \in \mathbf{N} \cup 0$ tels que $\lambda + \nu \leqslant s$ (c. à d. $g - s + 2\lambda + l - s + 2\nu \leqslant n$) il existe des polynômes $P \in H_{g,0,l}$, $Q \in H_{r,0,s}$ pour lesquels $R_{n,k}$ a exactement $g - s + 2\lambda$ racines simples strictement négatives et exactement $l - s + 2\nu$ racines simples strictement positives.*

Pour considérer le cas général où P et Q peuvent avoir des racines nulles il suffit de combiner Théorème 0.3 avec Théorème 0.2.

1. English version

The present Note is a continuation of paper [2] (which was the result of a fruitful collaboration with B.Z. Shapiro). We consider real polynomials in one real variable of the form $P(x) = \sum_{j=0}^{n} C_n^j a_j x^j$. Set $Q(x) = \sum_{j=0}^{n} C_n^j b_j x^j$. The *composition of Schur–Szegö* of P and Q is defined as $P {}_n^* Q = \sum_{j=0}^{n} C_n^j a_j b_j x^j$. Throughout the paper one has $k \in \mathbf{N} \cup 0$.

Remark 1. If P and Q are considered as polynomials of degree $n + 1$ with leading coefficients 0, then $P * Q$ should be defined as $\sum_{j=0}^{n+1} ((C_n^j)^2 / C_{n+1}^j) a_j b_j x^j$ (setting $a_{n+1} = b_{n+1} = 0$) which is a different formula. The index n under $*$ is put to avoid such a possible ambiguity. One could think of $P {}_{n+k}^{*} Q$ as of the composition of two polynomials each of which has a k-fold root at ∞.

Example 1. One checks directly that $(P {}_n^* Q)' = \frac{1}{n}(P' {}_{n-1}^{*} Q')$. This formula is also valid when the k first coefficients of one or both polynomials P, Q are 0. Set $n \mapsto n + k$ and set (for the rest of the paper) $R_{n,k} := P {}_{n+k}^{*} Q$. Thus one has $R'_{n,k} = \frac{1}{n+k}(P' {}_{n+k-1}^{*} Q')$ for all k.

In this Note, in the case when P and Q are *hyperbolic*, i.e. with real roots only, we give the exhaustive answer (see Theorems 1.5 and Remarks 4, 8) to the question if the numbers of positive, negative and zero roots of P and Q are known, what these numbers can be for $R_{n,k}$.

Definition 1.1. A *multiplicity vector* (*MV*) is a vector whose components equal the multiplicities of the roots of a hyperbolic polynomial listed in the increasing order. Denote by Hyp_n the set of hyperbolic polynomials of degree n and by $H_{u,v,w}$ its subset of polynomials with u negative, w positive roots (counted with multiplicity) and a v-fold root at 0, $u+v+w=n$.

One has (see [2], Proposition 1.5)

$$H_{u,v,w} {}^{*}_{n} H_{n,0,0} \subset H_{u,v,w}. \tag{1}$$

Composition of polynomials in $H_{n,0,0}$ defines a semigroup action on the set of MVs (i.e. of ordered partitions of n), see [2], Proposition 1.4, Theorem 1.6 and Corollary 1.7.

Proposition 1.2.

(1) *If $R_{n,0} \in H_{u,v,w}$, then for all k one has $R_{n,k} \in H_{u,v,w}$ and $H_{u,v,w} {}^{*}_{n+k} H_{n,0,0} \subset H_{u,v,w}$. If $R_{n,0}$ is not necessarily hyperbolic and has u' negative, w' positive and a v'-fold root at 0, then for all k $R_{n,k}$ has $\geqslant u'$ negative, $\geqslant w'$ positive and a v'-fold root at 0.*
(2) *If $x_P \neq 0$ and $x_Q \neq 0$ are roots of P and Q of multiplicities m_P and m_Q, $m_P + m_Q \geqslant n+k$, then $-x_P x_Q$ is a root of $R_{n,k}$ of multiplicity $m_P + m_Q - n - k$.*

Indeed, one has $R_{n,k} = \sum_{j=0}^{n} C_n^j a_j b_j (C_n^j / C_{n+k}^j) x^j$, and $C_n^j / C_{n+k}^j = (n!/(n+k)!)(n+k-j)\cdots(n+1-j)$. Hence, $R_{n,k}$ is obtained from $n! R_{n,0}/(n+k)!$ by the following operations: (1) reverting, i.e. $R_{n,0}(x) \mapsto x^n R_{n,0}(1/x)$ (the monomial $C_n^j a_j b_j x^j$ changes to $C_n^j a_j b_j x^{n-j}$); (2) multiplication by x^k; (3) k-fold differentiation; (4) reverting. Each of these operations doesn't decrease the number of positive and negative roots counted with multiplicity; the multiplicity of 0 as a root doesn't change; in particular, if $R_{n,0} \in H_{u,v,w}$, then $R_{n,k} \in H_{u,v,w}$. Part (2) for $k=0$ is Proposition 1.4 of [2], for $k>0$ it follows from the k-fold differentiation in (3).

Remark 2. When $R_{n,0} \in Hyp_n$, then the MV of $R_{n,0}$ defines the one of $R_{n,k}$ for all $k>0$. This follows from operations (1)–(4) used in the above proof, from the Rolle theorem and from $R_{n,k} \in Hyp_n$ for all k, see Lemma 4.2 from [3].

Remark 3. The polynomials P and Q are *apolar* if $\sum_{j=0}^{n} (-1)^j C_n^j a_j b_{n-j} = 0 (*)$ (see more about apolarity in [4]). Suppose that $m_P + m_Q > n$ and $x_P = x_Q \neq 0$ (see part (2) of Proposition 1.2). Set $Q_1 := x^n Q(1/x)$. Hence, $1/x_Q$ is a root of Q_1 of multiplicity m_Q. By Proposition 1.4 from [2], $-x_P/x_Q = -1$ is a root of $P {}^{*}_{n} Q_1$, i.e. $(*)$ holds and P, Q are apolar.

Theorem 1.3. *Suppose that P and Q are complex polynomials of degree n such that $Q = x^q S$, $\deg(S) = n-q$. Then for all k one has $P {}^{*}_{n+k} Q = \frac{(n+k-q)!}{(n+k)!} x^q (P^{(q)} {}^{*}_{n+k-q} S)$.*

Indeed, for $k=0$ one has $R_{n,0} = x^q \sum_{j=0}^{n-q} C_n^{j+q} a_{j+q} b_{j+q} x^j$ and

$$x^q (P^{(q)} {}^{*}_{n-q} S) = x^q \sum_{j=0}^{n-q} \frac{(j+q)!}{j!} \frac{C_n^{j+q} a_{j+q} C_n^{j+q} b_{j+q}}{C_{n-q}^j} x^j = \frac{n!}{(n-q)!} x^q \sum_{j=0}^{n-q} C_n^{j+q} a_{j+q} b_{j+q} x^j.$$

For $k>0$ just consider P and Q as polynomials of degree $n+k$ with k leading zero coefficients.

Remark 4. When $P \in H_{u,v,w}$, $Q \in H_{n-q,q,0}$, $q \geqslant 1$, the theorem shows that the MV of $R_{n,0}$ is not defined by the MVs of P and Q, but by the ones of $P^{(q)}$ and Q. In this case one has $R_{n,0} \in Hyp_n$ (this follows from (1) in the limit when q of the roots of Q tend to 0).

Denote by $x_j^{(i)}$ the roots of $P^{(i)}$, $i=0,\dots,n-1$, $j=1,\dots,n-i$, $x_j^{(i)}\leqslant x_{j+1}^{(i)}$. Set $x_j=x_j^{(0)}$. If $v=0$, then the Rolle theorem yields $x_{u-q}^{(q)}\leqslant x_u$ and $x_{u+1}\leqslant x_{u+1}^{(q)}$ whenever $x_{u-q}^{(q)}$ and/or $x_{u+1}^{(q)}$ are meaningful. One can have in particular $x_u<x_{u-q+1}^{(q)}<x_u^{(q)}<x_{u+1}$, see [3], Theorem 4.4. Hence the number 0 is either in one of the intervals $(x_s^{(q)},x_{s+1}^{(q)})$, $s=u-q+1,\dots,u-1$, or $(x_u,x_{u-q+1}^{(q)})$, or $(x_u^{(q)},x_{u+1})$, or equals $x_s^{(q)}$ for $s=u-q+1,\dots,u$. Thus one can have $R_{n,0}\in H_{h,q,n-h-q}$ for $h=u-q,\dots,u$ or $R_{n,0}\in H_{h,q+1,n-h-q-1}$ for $h=u-q,\dots,u-1$. If $v>0$, then set $m=\min(v,q)$. The theorem implies that

$$R_{n,0}=\frac{(n-m)!}{n!}x^m\bigl(P^{(m)}{}_{n-m}^{*}(x^{q-m}S)\bigr). \tag{2}$$

For $m=v\leqslant q$ in the same way one sees that either $R_{n,0}\in H_{h,q,n-h-q}$ for $h=u-q+v,\dots,u$ or $R_{n,0}\in H_{h,q+1,n-h-q-1}$ for $h=u-q+v,\dots,u-1$. For $v>q=m$ it follows from (2) that $R_{n,0}\in H_{u,v,w}$ because $S\in H_{n-q,0,0}$, $P^{(m)}\in H_{u,v-q,w}$ and $P^{(m)}{}_{n-q}^{*}S\in H_{u,v-q,w}$, see (1).

Remark 5. One has $P(\alpha x)_{n+k}^{*}Q(\beta x)=R_{n,k}(\alpha\beta x)$ (for all k and for all $\alpha,\beta\in\mathbf{R}^*$).

Definition 1.4. For a degree n complex polynomial P as above set $A_\zeta P:=(\zeta-x)P'+nP$ (the polar derivative of P w.r.t. the point $\zeta\neq\infty$; for $\zeta=\infty$ one sets $A_\zeta P:=P'$). Hence, one has $A_0P(x)=\sum_{j=0}^n(n-j)C_n^ja_jx^j$.

Remark 6. If P is hyperbolic, then so is $A_\zeta P$ – $\operatorname{sign}(A_\zeta P)$ changes alternatively at the roots of P'. As $\deg(A_\zeta P)\leqslant n-1$ with equality for $\zeta\neq-a_{n-1}/a_n$, we set

$$A_0(A_0P)=-x(A_0P)'+(n-1)A_0P=x^2P''-2(n-1)xP'+n(n-1)P''.$$

Set $P^{[j]}=A_0(A_0(\cdots A_0P)\cdots)$ (j times A_0). Consider P as a polynomial of degree $n+k$ with $a_i=0$ for $i=n+1,\dots,n+k$. Set $A_{0,k}P=(n+k)P-xP'=P^{[1,k]}=\sum_{j=0}^n(n+k-j)C_n^ja_jx^j$ and $P^{[j,k]}=A_{0,k}(A_{0,k}(\cdots A_{0,k}P)\cdots)$ (j times). One has

$$n!(P_n^*Q)(-x^2)=\sum_{j=0}^n(-1)^jx^{n-j}P^{[j]}(x)Q^{(n-j)}(x). \tag{3}$$

Indeed, one has $P^{[j]}=\sum_{\nu=0}^{n-j}C_n^\nu\frac{(n-\nu)!}{(n-j-\nu)!}a_\nu x^\nu$, $Q^{(n-j)}=\sum_{\nu=n-j}^nC_n^\nu\frac{\nu!}{(\nu-n+j)!}b_\nu x^{\nu-(n-j)}$ and the coefficient before x^s in the right hand-side of (3) equals $g_s:=\sum_{j=0}^n(-1)^j\sum_{\nu=0}^sa_{s-\nu}b_\nu\eta_{j,\nu}$ where

$$\eta_{j,\nu}=C_n^{s-\nu}\frac{(n-s+\nu)!}{(n-j-s+\nu)!}C_n^\nu\frac{\nu!}{(\nu-n+j)!}=p_{s,\nu}C_{2\nu-s}^{\nu-n+j},\quad p_{s,\nu}=\frac{(n!)^2}{(n-\nu)!(s-\nu)!(2\nu-s)!}$$

(meaningless $\eta_{j,\nu}$ are set to be 0). Thus the coefficient before $a_{s-\nu}b_\nu$ in g_s equals $p_{s,\nu}\sum_{j=n-\nu}^{\nu-s+n}(-1)^jC_{2\nu-s}^{\nu-n+j}=p_{s,\nu}\delta_{\nu,s-\nu}(-1)^{n-\nu}=(-1)^{n-\nu}n!C_n^\nu$ if $\nu=s-\nu$, i.e. $s=2\nu$, and 0 if not.

One checks directly that for $k\geqslant1$ and for $k=a_n=0$ one has

$$P_{n+k}^{*}Q=\sum_{j=0}^{n-1}\frac{C_n^ja_jC_n^jb_jx^j}{C_{n+k}^j}=\frac{1}{n+k}\bigl(P_{n+k-1}^{*}Q^{[1,k]}\bigr)=\sum_{j=0}^{n-1}\frac{C_n^ja_j(n+k-j)C_n^jb_jx^j}{(n+k)C_{n+k-1}^j}. \tag{4}$$

Remark 7. The composition of Schur–Szegö is associative and commutative. Further we write $[PQR]_n$ for $P_n^*Q_n^*R$ etc. It is easy to show that every degree n polynomial having one of its roots at (-1) (by Remark 5 this assumption is not 'too restrictive') is representable in the form $\tilde K:=[K_{a_1}\cdots K_{a_{n-1}}]_n$ where $K_a:=(x+1)^{n-1}(x+a)$. This representation is unique modulo permutation of the a_i. The dependence of the numbers a_j on the roots of the polynomial S is not trivial. E.g. for the polynomial $(x+1)x^{n-1}$ (with an $(n-1)$-fold zero at 0) one has $a_j=-\frac{j-1}{n-j+1}$, $j=1,\dots,n-1$, i.e. only a_1 is 0. (Up to permutation of the indices j, the coefficient before x^{j-1} of K_{a_j} must equal 0.) For $n=3$ set $U_\varepsilon:=(x+1)^2(x-\varepsilon)$. Hence $U_0{}_3^*U_0=(x+1)(x+1/3)x$. For $a,b\in\mathbf{R}$ small enough $U_{a+bi}{}_3^*U_{a-bi}$

has three distinct negative roots. For $a_1 = \cdots = a_{n-1} = a > 1$ all roots of $\tilde{K}$ are simple and negative. This can be deduced from Proposition 1.4 and Theorem 1.6 of [2]. See more details about $\tilde{K}$ in [1].

Consider the case $P \in H_{g,0,l}$, $Q \in H_{r,0,s}$. One can assume without loss of generality that $g \geqslant l$ and $s = \min(g,l,r,s)$ (if necessary use Remark 5 with $\alpha = -1$ and/or $\beta = -1$).

Theorem 1.5.

(1) *Under these assumptions for any k, $R_{n,k}$ has $\geqslant g-s$ negative and $\geqslant l-s$ positive roots counted with multiplicity, and $\leqslant s$ complex conjugate couples.*
(2) *For all k and for all $\lambda, \nu \in \mathbf{N} \cup 0$ such that $\lambda + \nu \leqslant s$ (i.e. $g - s + 2\lambda + l - s + 2\nu \leqslant n$) there exist polynomials $P \in H_{g,0,l}$, $Q \in H_{r,0,s}$ for which $R_{n,k}$ has exactly $g - s + 2\lambda$ negative and exactly $l - s + 2\nu$ positive simple roots.*

Remark 8. The theorem is inspired by the following example: if $P = (x+\alpha)^g(x-\beta)^l$, $Q = (x+\gamma)^r(x-\delta)^s$, $\alpha, \beta, \gamma, \delta > 0$, then by Proposition 1.4 from [2], $R_{n,0}$ has roots $-\alpha\gamma$ and $\beta\gamma$, of multiplicities respectively $g + r - n = g - s$ and $l + r - n = l - s$. One can extend Theorem 1.5 to the case when P and Q can have roots at 0 by means of Theorem 1.3.

Proof of Theorem 1.5.
1^0. The theorem (parts (1) and (2)) is checked directly for $n = 1$ and any k.

2^0. Prove directly part (2) in Case A: n is even, $g = l = r = s = n/2$ and $\lambda = \nu = 0$. Set $P = Q = (x^2 - 1)^{n/2}$. Hence, $R_{n,k}$ contains only even powers of x and their coefficients are > 0, i.e. $R_{n,k}$ has no real roots. Further for even n we assume that if $g = l = r = s = n/2$, then $\lambda \geqslant \nu$, otherwise use Remark 5 with $\alpha = -1$, $\beta = 1$ to exchange λ and ν.

3^0. We prove part (2) by induction on n in 3^0–5^0. We deduce the claim for (n,k) from the one for $(n-1, k+1)$. Denote by $P \in H_{g-1,0,l}$ and $Q \in H_{r-1,0,s}$ two monic polynomials of degree $n-1$ for which $R_{n-1,k+1} := P {}_{n+k}^{*} Q$ has exactly $g - 1 - s + 2\lambda$ negative and $l - s + 2\nu$ positive simple roots. Denote all the roots of $R_{n-1,k+1}$ by ζ_i where $\zeta_i \in \mathbf{R}$ for $1 \leqslant i \leqslant n - 1 + 2\lambda + 2\nu - 2s$. One has always $l - s + 2\nu \geqslant 0$. One has $g - 1 - s + 2\lambda < 0$ only when n is even and $g = l = r = s = n/2$, $\lambda = 0$. As $\lambda \geqslant \nu$, see 2^0, one has $\lambda = \nu = 0$ and this is Case A which was considered in 2^0.

4^0. Consider the polynomial $T := x^{n-1}(x+1)$ as limit for $\varepsilon \to 0$, $\varepsilon > 0$, of each of the two one-parameter families $P_\varepsilon := \varepsilon^{n-1} P(x/\varepsilon)(x+1)$ and $Q_\varepsilon := \varepsilon^{n-1} Q(x/\varepsilon)(x+1)$. One has $\tilde{T} := T {}_{n+k}^{*} T \in H_{1,n-1,0}$ for any k. The negative root of $\tilde{T}$ equals $-(k+1)/n$. Hence, for $\varepsilon > 0$ small enough the polynomial $U_\varepsilon(x) := P_\varepsilon(x) {}_{n+k}^{*} Q_\varepsilon(x)$ has a negative simple root ξ close to $-(k+1)/n$ and $n-1$ roots close to 0.

5^0. Set $x \mapsto \varepsilon x$. Hence, P_ε, Q_ε become perturbations of P, Q, the root ξ of U_ε becomes ξ/ε, its roots close to 0 equal $\zeta_i + o(\varepsilon)$. For small $\varepsilon > 0$ they are real, simple, different from ξ/ε and close to ζ_i, so U_ε has exactly $g - s + 2\lambda$ negative, $l - s + 2\nu$ positive and $2(s - \lambda - \nu)$ complex roots. Part (2) of the theorem is proved.

6^0. To prove part (1) of the theorem it suffices to consider the case $k = 0$ and when all roots of P and Q are simple. For $k \geqslant 1$ the result would follow from part (1) of Proposition 1.2; it will be extended by continuity to the case when P and/or Q has multiple roots. Suppose first that $s = 1$. Set $Q = (x - c)G$ where $c > 0$ and all roots of G are negative. One has

$$V(x) := \left((x-c)G {}_n^{*} P\right) = \left((xG) {}_n^{*} P\right) - c(G {}_n^{*} P) = \frac{x}{n}(G {}_{n-1}^{*} P') - \frac{c}{n}\left(G {}_{n-1}^{*} P^{[1]}\right) \tag{5}$$

see Theorem 1.3 with $k = 0$, $q = 1$, and equalities (4) with $k = a_n = 0$. Observe that $P^{[1]} + \zeta P' = A_\zeta P$, see Definition 1.4. Consider the degree $n-1$ polynomial $A_{(-\lambda/c)} P$. Set $W(x,\lambda) := (-c/n)(G {}_{n-1}^{*} A_{(-\lambda/c)} P)$. Hence (see (5)) one has $V(x) = W(x,x)$.

7^0. The polynomial $A_{(-\lambda/c)} P$ has (for every $\lambda \neq c a_{n-1}/a_n$ fixed) $n-1$ real simple roots depending smoothly on λ. (For $\lambda = c a_{n-1}/a_n$ one has $\deg A_{(-\lambda/c)} P < n - 1$, i.e. some roots go to ∞.) The same is true for $W(x,\lambda)$ when considered as a polynomial in x. Hyperbolicity of W follows from (1) and $G \in H_{n-1,0,0}$. Simplicity of its roots follows from Theorem 1.6 in [2]; simple roots depend smoothly on parameters.

8^0. Denote by $x_1 < \cdots < x_g < 0 < x_{g+1} < \cdots < x_n$ and $y_1 < \cdots < y_{n-1}$ the roots of P and P'. For all $\lambda \in \mathbf{R}$ one has $\operatorname{sign}(A_{(-\lambda/c)} P(y_j)) = (-1)^{n-j}$. Therefore for all λ, $A_{(-\lambda/c)} P$ has $\geqslant g - 2$ distinct roots in (y_1, y_{g-1}), $\geqslant l - 2$

distinct roots in (y_{g+1}, y_{n-1}) and a root in (y_{g-1}, y_{g+1}). By (1) the same is true for $W(x,\lambda)$. Denote the roots of $W(x,\lambda)$ by $\psi_j(\lambda)$.

9^0. For $|\lambda|$ large enough ($|\lambda| \geqslant M$), $A_{(-\lambda/c)}P$ has $n-1$ simple real roots which are close to the ones of P'. For such λ all roots of $W(x,\lambda)$ belong to $[-Nx^*, Nx^*]$ where $x^* = \max(|x_1|, |x_n|)$ and N is the maximal of the modules of roots of G, see Proposition 1.2 in [2]. Suppose that $M \geqslant Nx^*$.

10^0. When λ varies in $[-M, M]$, it takes $\geqslant g+l-4$ values $\lambda_1 < \cdots < \lambda_{g-2} < 0 < \lambda_{g+1} < \cdots < \lambda_{n-2}$ for which one has $W(\lambda_j, \lambda_j) = 0$ (apply the Bolzano theorem to the functions $\lambda - \psi_j(\lambda)$ on $[-M, M]$). Hence, for $x = \lambda_j$ one has $W(x,x) = V(x) = 0$, i.e. $V(x)$ has $\geqslant g-2$ distinct negative and $\geqslant l-2$ distinct positive roots.

11^0. One has $\operatorname{sign}(V(0)) = \operatorname{sign}(P(0))\operatorname{sign}(Q(0)) = -\operatorname{sign}(P(0)) = (-1)^{n+g-1}$. If $V(x)$ has exactly $g-2$ negative roots, then $\operatorname{sign}(V(0)) = (-1)^{n+g-2}$. Hence, $V(x)$ has $\geqslant g-1$ negative roots. Using Remark 5 with $\alpha = -1$, $\beta = 1$, one changes the signs of the roots and finds that $V(-x)$ (resp. $V(x)$) has $\geqslant l-1$ negative (resp. positive) roots.

12^0. Prove part (1) for $s > 1$. Denote by $c_1, \ldots, c_s$ the positive roots of Q. Set $P_0 = P$, $P_j = A_{(-\lambda/c_j)}P_{j-1}$, $j = 1, \ldots, s$. For $j \geqslant 1$ the polynomials P_j depend on λ. One has $\deg(P_j) \leqslant n-j$ with equality for all except finitely many real values of λ, see 7^0.

Set $Q = (x-c_1)\cdots(x-c_j)G_j$, $1 \leqslant j \leqslant s$, and for $j = 1, \ldots, s$ set

$$V_j(x,\lambda) = \left((x-c_j)G_j {}_{n-j+1}^{\;*} P_{j-1}\right) = \frac{x}{n-j+1}\left(G_j {}_{n-j}^{\;*} P'_{j-1}\right) - \frac{c_j}{n-j+1}\left(G_j {}_{n-j}^{\;*} P_{j-1}^{[1]}\right) \quad \text{and}$$

$$W_j(x,\lambda) = -\frac{c_j}{n-j+1}\left(G_j {}_{n-j}^{\;*} A_{(-\lambda/c_j)}P_{j-1}\right).$$

Applying s times the reasoning from 8^0 one shows that for all λ, P_s has $\geqslant g-s-1$ distinct roots in (y_1, y_{g-1}) and $\geqslant l-s-1$ distinct roots in (y_{g+1}, y_{n-1}). These roots depend smoothly on λ. For $|\lambda|$ large enough ($|\lambda| \geqslant M_s$, M_s is defined by analogy with M, see 9^0) the roots of P_s are close to the ones of $P^{(s)}$. When λ varies in $[-M_s, M_s]$, it takes $\geqslant g-s-1$ distinct negative and $\geqslant l-s-1$ positive values λ_j for which one has $W_s(\lambda_j, \lambda_j) = 0$. Hence, for $x = \lambda_j$ one has $W_s(x,x) = V_s(x,x) = 0$, i.e. V_s has $\geqslant g-s-1$ negative and $\geqslant l-s-1$ positive roots. As in 11^0 one concludes that the negative (resp. positive) roots are $\geqslant g-s$ (resp. $\geqslant l-s$). □

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