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Fluctuation of stochastic systems with average equilibrium point

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Abstract

This Note deals with the diffusion approximation of dynamical systems where the velocity function contains a perturbing term. Thus, the problem considered is the investigation of the fluctuation of the initial system with respect to the above perturbing term. ***To cite this article: Y. Chabaniuk et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Fluctuation des systèmes stochastiques avec point d'équilibre en moyenne. Cette Note concerne l'approximation de diffusion des systèmes dynamiques où la fonction de vitesse contient un terme de perturbation. Ainsi le problème considéré est l'étude de la fluctuation du système initial par rapport au terme de perturbation. ***Pour citer cet article : Y. Chabaniuk et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

Le système stochastique considéré dans cette Note est donné par une solution de l'équation :

$$\frac{\mathrm{d}u^\varepsilon}{\mathrm{d}t}(t) = C\big(u^\varepsilon; x(t/\varepsilon^4)\big) + \varepsilon^{-1} C_0\big(x(t/\varepsilon^4)\big). \tag{a}$$

Le terme de perturbation considéré ici est donné sous la forme suivante :

$$C_0^\varepsilon(t) = \varepsilon^{-2} \int_0^t C_0\big(x(s/\varepsilon^4)\big)\,\mathrm{d}s. \tag{b}$$

L'échelle du temps ε^{-4} est choisi afin d'éviter de puissances non entières de ε.

Le processus $x(t), t \geqslant 0$, est Markovien ergodique avec générateur $\mathbf{Q}$ et loi stationnaire π.

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Le processus de fluctuation est considéré sous la forme suivante :

$$V^\varepsilon(t) = \big[u^\varepsilon(t) - \varepsilon C_0^\varepsilon(t)\big]/\varepsilon. \tag{c}$$

De (a) et (b) on obtient :

$$\frac{\mathrm{d}V^\varepsilon}{\mathrm{d}t}(t) = \varepsilon^{-1} C\big(u^\varepsilon(t); x(t/\varepsilon^4)\big), \tag{d}$$

avec

$$u^\varepsilon(t) = \varepsilon\big[V^\varepsilon(t) + C_0^\varepsilon(t)\big]. \tag{f}$$

Les relations (d) et (f) sont utilisées afin de démontrer le théorème suivant.

La relation d'équilibre suivante :

$$\int_E \pi(\mathrm{d}x) C_0(x) = 0, \tag{g}$$

et l'existance d'un point d'équilibre de la vitesse moyenne

$$\widehat{C}(u) = \int_E \pi(\mathrm{d}x) C(u; x),$$

soit

$$\widehat{C}(0) = 0, \tag{h}$$

sont essentielles.

Théorème 0.1. *Sous la condition d'équilibre* (g) *et l'existance d'un point d'équilibre unique de la vitesse moyenne* (h), *on a la convergence faible suivante* :

$$\big(V^\varepsilon(t), C_0^\varepsilon(t)\big) \Rightarrow \big(\zeta(t), W_\sigma(t)\big), \quad \varepsilon \to 0.$$

Le processus limite $\zeta(t), W_\sigma(t), t \geqslant 0$ est définit par le générateur

$$\mathbf{L}\varphi(v, w) = (v + w) c \varphi_v'(v, w) + \frac{1}{2} B \varphi_w''(v, w),$$

où $c = \widehat{C}_u'(0)$ est la dérivée au point $u = 0$ de la vitesse moyenne, et B est la matrice de dispersion donnée par

$$B = 2\int_E \pi(\mathrm{d}x) C_0(x) \mathbf{R}_0 C_0(x),$$

avec $\mathbf{R}_0$ l'operateur potentiel du processus markovien $x(t), t \geqslant 0$.

C'est-à-dire que le processus limite $\zeta(t)$ satisfait l'équation différentielle stochastique

$$\mathrm{d}\zeta(t) = \big(\zeta(t) + W_\sigma(t)\big) c \, \mathrm{d}t.$$

1. Introduction and preliminaries

Stochastic dynamical systems are used in different applied fields under different conditions (see, e.g., [1–4]).

The stochastic system with Markov switching on the series scheme, with the small series parameter $\varepsilon > 0$ ($\varepsilon \to 0$), is defined as a solution of the evolutionary equation (see [1, ch. 3])

$$\frac{\mathrm{d}u^\varepsilon}{\mathrm{d}t}(t) = C\big(u^\varepsilon(t); x(t/\varepsilon^2)\big) + \varepsilon^{-1} C_0\big(x(t/\varepsilon^2)\big). \tag{1}$$

These kinds of time-scaled systems are studied in [1], where diffusion approximation is obtained.

The velocity function $C(u; x), u \in \mathbb{R}^d, x \in E$, satisfies the conditions of a global solution of the associated deterministic evolutionary equations

$$\frac{\mathrm{d}u_x}{\mathrm{d}t}(t) = C\big(u_x(t); x\big), \quad x \in E. \tag{2}$$

The switching Markov process $x(t), t \geqslant 0$, on the standard phase space $(E, \mathcal{E})$, (E is a Polish space, and $\mathcal{E}$ its Borel σ-field), is given by the generator (see [1, ch.1])

$$\mathbf{Q}\varphi(x) = q(x) \int_E P(x, \mathrm{d}y)\big[\varphi(y) - \varphi(x)\big], \quad \varphi \in \mathcal{B}(E), \tag{3}$$

where $\mathcal{B}(E)$ is the Banach space, of all measurable bounded functions defined on E, $q(x)$ is the jump intensity function, and $P(x, \mathrm{d}y)$ the transition kernel of the embedded Markov chain $x_n = x(\tau_n), n \geqslant 0$, with the jump times $0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots$. The Markov process $x(t), t \geqslant 0$, is supposed to be uniformly ergodic with stationary distribution $\pi(B), B \in \mathcal{E}$. Let us denote by $\mathbf{R}_0$ the potential operator of $\mathbf{Q}$, that is, $\mathbf{R}_0 = \Pi - (\Pi + \mathbf{Q})^{-1}$, and $\Pi(x, \mathrm{d}y) = \pi(\mathrm{d}y)$.

The average system is defined by a solution of the average equation

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}t}(t) = \widehat{C}\big(\hat{u}(t)\big), \tag{4}$$

with the average velocity

$$\widehat{C}(u) = \int_E \pi(\mathrm{d}x) C(u; x). \tag{5}$$

The perturbing term in (1), under the balance condition

$$\int_E \pi(\mathrm{d}x) C_0(x) = 0, \tag{6}$$

converges weakly (see [1, ch. 3])

$$\varepsilon^{-1} \int_0^t C_0\big(x(s/\varepsilon^2)\big)\,\mathrm{d}s \Rightarrow W_\sigma(t), \quad \varepsilon \to 0. \tag{7}$$

Remark 1. In fact a balance condition, like condition (6), is necessary for every central limit theorem.

The limit Wiener process $W_\sigma(t), t \geqslant 0$, is defined by the covariance matrix $B = \sigma\sigma^*$ as follows [1, ch. 3]

$$B = 2 \int_E \pi(\mathrm{d}x) C_0(x) \mathbf{R}_0 C_0(x). \tag{8}$$

We consider here the problem of investing the fluctuation of the initial stochastic system with respect to the perturbing term under the condition of the existence of a unique equilibrium point of the average velocity

$$\widehat{C}(0) = 0. \tag{9}$$

Remark 2. From the stability theory, it is well known that the point $u = 0$ is the unique equilibrium point of the average velocity $\widehat{C}(u)$ (9), which under additional stability conditions allows for the weak convergence to zero of the solution of the average equation (4).

We assume also that there exists bounded function $C'_u(0; x) =: C_1(x)$, where $C'_u(0; x)$ is the derivative of function $C(u; x)$ with respect to the u variable at point $u = 0$.

2. Results

The initial stochastic system considered here is given by a solution of the equation

$$\frac{\mathrm{d}u^\varepsilon}{\mathrm{d}t}(t) = C\big(u^\varepsilon(t); x(t/\varepsilon^4)\big) + \varepsilon^{-1} C_0\big(x(t/\varepsilon^4)\big) \tag{10}$$

the perturbation term considered here is given in the following form:

$$C_0^\varepsilon(t) = \varepsilon^{-2} \int_0^t C_0\big(x(s/\varepsilon^4)\big)\,\mathrm{d}s. \tag{11}$$

Remark 3. Clearly, Eq. (10) is not the same as Eq. (1). In the case of Eq. (1) we can get diffusion approximation results (see [1]) without scaling as we do in the case of Eq. (10). The time-scaling in (10) is used to avoid non-integer degree of ε.

It is worth noticing that under the balance condition (6) weak convergence

$$C_0^\varepsilon(t) \Rightarrow W_\sigma(t), \quad \varepsilon \to 0 \tag{12}$$

takes place. In fact, this is the same as in (7).

The fluctuation process is considered in the following form:

$$V^\varepsilon(t) = \big[u^\varepsilon(t) - \varepsilon C_0^\varepsilon(t)\big]/\varepsilon. \tag{13}$$

From (10) and (11) we get

$$\frac{\mathrm{d}V^\varepsilon}{\mathrm{d}t}(t) = \varepsilon^{-1} C\big(u^\varepsilon(t); x(t/\varepsilon^4)\big), \tag{14}$$

with

$$u^\varepsilon(t) = \varepsilon\big[V^\varepsilon(t) + C_0^\varepsilon(t)\big]. \tag{15}$$

These two relations (14) and (15) are used to prove the following:

Theorem 2.1. *Under the balance condition* (6) *and the existence of the unique equilibrium point* (9) *we have the following weak convergence*

$$\big(V^\varepsilon(t), C_0^\varepsilon(t)\big) \Rightarrow \big(\zeta(t), W_\sigma(t)\big), \quad \varepsilon \to 0. \tag{16}$$

The limit process $\zeta(t), W_\sigma(t), t \geqslant 0$ is defined by the generator

$$\mathbf{L}\varphi(v, w) = (v + w)c\varphi_v'(v, w) + \frac{1}{2}B\varphi_w''(v, w), \tag{17}$$

where $c = \widehat{C}_u'(0)$ and the covariance matrix B is defined in (8). *That is the limit process $\zeta(t)$ satisfies the stochastic differential equation*

$$\mathrm{d}\zeta(t) = \big(\zeta(t) + W_\sigma(t)\big)c\,\mathrm{d}t. \tag{18}$$

Remark 4. The limit process can be represented as follows:

$$\zeta(t) = \eta(t) - W_\sigma(t),$$

where

$$\mathrm{d}\eta(t) = c\eta(t)\,\mathrm{d}t + \mathrm{d}W_\sigma(t).$$

3. Proof of theorem

The random evolution approach with a solution of the singular perturbation problem will be used (see [1, ch. 4–6]).

Lemma 3.1. *The three-component Markov process*

$$V^\varepsilon(t), \quad C_0^\varepsilon(t), \quad x_t^\varepsilon := x(t/\varepsilon^4), \quad t \geqslant 0,$$

can be characterized by the generator

$$\mathbf{L}^{\varepsilon}\varphi(v,w,x)=\left[\varepsilon^{-4}\mathbf{Q}+\varepsilon^{-2}\mathbf{C}_0(x)+\varepsilon^{-1}\mathbf{C}(x)+\mathbf{C}_1(x)+\varepsilon\theta^{\varepsilon}(x)\right]\varphi, \tag{19}$$

where

$$\begin{aligned}&\mathbf{C}_0(x)\varphi(w)=C_0(x)\varphi'(w),\\&\mathbf{C}(x)\varphi(v)=C(0,x)\varphi'(v),\\&\mathbf{C}_1(x)\varphi(v)=(v+w)C'_u(0,x)\varphi'(v),\end{aligned} \tag{20}$$

and the negligible term

$$\left\|\theta^{\varepsilon}(x)\varphi\right\|\to 0,\quad \varepsilon\to 0,\ \varphi(v)\in C^2(\mathbb{R}^d). \tag{21}$$

Here we put just the argument of φ where the corresponding operator acts.

Proof. Let us calculate the conditional expectation using Eqs. (11), (14), (15)

$$\begin{aligned}&E\left[\varphi\left(v+\Delta V^{\varepsilon}(t),w+\Delta C_0^{\varepsilon}(t),x^{\varepsilon}_{t+\Delta}\right)-\varphi(v,w,x)\;\middle|\;x^{\varepsilon}_t=x\right]\\&\quad=E_x\left[\varphi(v,w,x^{\varepsilon}_{t+\Delta})-\varphi(v,w,x)+E_x\left[\Delta C\left(\varepsilon(v+w),x\right)\varphi'_v(v,w,x)+\varepsilon^{-2}\Delta C_0(x)\varphi'_w(v,w,x)\right]\right]+\mathrm{o}(\Delta).\end{aligned}$$

So, the generator $\mathbf{L}^{\varepsilon}$ is

$$\mathbf{L}^{\varepsilon}\varphi(v,w,x)=\varepsilon^{-2}\mathbf{Q}\varphi+\varepsilon^{-1}C\left(\varepsilon(v+w);x\right)\varphi'_v+\varepsilon^{-2}C_0(x)\varphi'_w. \tag{22}$$

The asymptotic representation (19) is obtained by using the Taylor formula for $C(u;x)$. □

Lemma 3.2. *A solution of the singular perturbation problem for the truncated generator*

$$\mathbf{L}_0^{\varepsilon}\varphi(v,w,x)=\varepsilon^{-4}\mathbf{Q}+\varepsilon^{-2}\mathbf{C}_0(x)+\varepsilon^{-1}\mathbf{C}(x)+\mathbf{C}_1(x) \tag{23}$$

is given in the following formulae:

$$\mathbf{L}_0^{\varepsilon}\varphi^{\varepsilon}=\mathbf{L}_0^{\varepsilon}\left[\varphi(v,w)+\varepsilon^2\varphi_2(v,w,x)+\varepsilon^3\varphi_1(v,w,x)+\varepsilon^4\varphi_0(v,w,x)\right]=\mathbf{L}\varphi(v,w)+\varepsilon\theta^{\varepsilon}(x)\varphi(v,w), \tag{24}$$

with the negligible term

$$\left\|\theta^{\varepsilon}(x)\varphi\right\|\to 0,\quad \varepsilon\to 0,\ \varphi(v,w)\in C^{3,3}(\mathbb{R}^d\times\mathbb{R}^d).$$

The limit operator is given by

$$\mathbf{L}\Pi=\Pi\mathbf{C}_1(x)\Pi+\Pi\mathbf{C}_0(x)\mathbf{R}_0\mathbf{C}_0(x)\Pi. \tag{25}$$

Proof. Following the method presented in [1, ch. 5] we calculate

$$\mathbf{L}_0^{\varepsilon}\varphi^{\varepsilon}=\varepsilon^{-4}\mathbf{Q}\varphi+\varepsilon^{-2}\left[\mathbf{Q}\varphi_2+\mathbf{C}_0(x)\varphi\right]+\varepsilon^{-1}\left[\mathbf{Q}\varphi_1+\mathbf{C}(x)\varphi\right]+\left[\mathbf{Q}\varphi_0+\mathbf{C}_0(x)\varphi_2\right]+\varepsilon\theta^{\varepsilon}(x)\varphi. \tag{26}$$

Now it is evident that

$$\mathbf{Q}\varphi(v,w)=0. \tag{27}$$

The balance condition (6) provides solvability condition for the equation

$$\mathbf{Q}\varphi_2+\mathbf{C}_0(x)\varphi=0, \tag{28}$$

here

$$\varphi_2=\mathbf{R}_0\mathbf{C}_0(x)\varphi. \tag{29}$$

The equilibrium point condition (9) provides solvability condition for the equation

$$\mathbf{Q}\varphi_1+\mathbf{C}(x)\varphi=0. \tag{30}$$

Here

$$\varphi_1 = \mathbf{R}_0 \mathbf{C}(x)\varphi. \tag{31}$$

At last the solvability condition for the equation

$$\mathbf{Q}\varphi_0 + \big[\mathbf{C}_1(x) + \mathbf{C}_0(x)\mathbf{R}_0\mathbf{C}_0(x)\big]\varphi = \mathbf{L}\varphi$$

gives the representation (25) of the limit operator $\mathbf{L}$.

It is easy to calculate the reminder term in (24) which is

$$\theta^\varepsilon(x)\varphi = \mathbf{L}_1^\varepsilon \varphi_2 + \mathbf{L}_0^\varepsilon[\varphi_1 + \varepsilon\varphi_0],$$

where

$$\mathbf{L}_1^\varepsilon := \mathbf{C}(x) + \varepsilon \mathbf{C}_1(x), \qquad \mathbf{L}_0^\varepsilon := \mathbf{C}_0(x) + \varepsilon^2 \mathbf{C}(x) + \varepsilon^2 \mathbf{C}_1(x).$$

The function φ_0 is given by

$$\varphi_0 = \mathbf{R}_0 \tilde{\mathbf{L}}(x)\varphi,$$

where

$$\tilde{\mathbf{L}}(x) := \mathbf{L} - \mathbf{L}(x), \quad \mathbf{L}(x) := \mathbf{C}_1(x) + \mathbf{C}_0(x)\mathbf{R}_0\mathbf{C}_0(x). \qquad \square$$

The proof of the theorem is finished by proving the tightness of the family of processes following the same arguments as in [1, ch. 6].

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