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Partial Differential Equations/Mathematical Problems in Mechanics

The regularity of solutions of the primitive equations of the ocean in space dimension three

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Abstract

In this Note, the global existence of strong solutions of the primitive equations for the ocean in space dimension 3 with the Dirichlet boundary condition is obtained. The method of the proof can be easily adapted to treat full primitive equations in a domain with a varying bottom topography. ***To cite this article: I. Kukavica, M. Ziane, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

La régularité des solutions des équations primitives de l'océan en dimension trois. Dans cette Note, on établie l'existence globale des solutions fortes des équations primitives de l'océan en dimension 3 pour des conditions aux limites de type Dirichlet. La méthode de démonstration s'adapte aisément au cas des équations primitives générales dans un domaine avec un fond de topographie variable. ***Pour citer cet article : I. Kukavica, M. Ziane, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

In this Note, we prove the global existence and uniqueness of solutions of the primitive equations of the ocean in a bounded domain with Dirichlet boundary conditions. These equations are the fundamental physical model in geophysical dynamics and oceanography [8,11]. The mathematical formulation of the primitive equations was initiated by J.-L. Lions, Temam, and Wang in [5–7]. They proved the global existence of weak solutions and they studied asymptotic and numerical properties of the solutions. The local existence of strong solutions for initial data in H^1 was proven in [10,2]. Recently, Cao and Titi proved in [1] the global existence of strong solutions for the primitive equations in the case of Neumann boundary conditions on the bottom and the top (see also [3]).

In this Note, we settle the case of (physical) Dirichlet boundary conditions. We provide the sketch of the proof for the case of Dirichlet boundary conditions on the bottom and the sides, and the Neumann boundary condition on the

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top for flat topography. For further details, other physical boundary conditions, and the non-flat boundary topography, see [4].

2. The regularity of the primitive equations

We address the existence and the uniqueness of strong solutions for the primitive equations of the ocean:

$$\frac{\partial u_k}{\partial t} - \nu\Delta u_k + \sum_{j=1}^{3} \partial_j(u_j u_k) + \partial_k p = f_k, \quad k = 1, 2, \tag{PE$_k$}$$

with the divergence free condition $\sum_{k=1}^{3} \partial_k u_k = 0$. Denote $v = (u_1, u_2)$ and $u = (u_1, u_2, u_3)$. The equations are derived from the 3D Navier–Stokes system under the hydrostatic approximation assumption. The differences with the 3D NSE are the lack of an evolution equation for u_3 and the fact that p is independent of x_3. The initial condition is $v(\cdot, 0) = v_0$, where $v_0 = (u_{01}, u_{02}) : \Omega \to \mathbb{R}^2$ satisfies $\mathrm{div}_2 \int_{-h}^{0} v_0 \,\mathrm{d}x_3 = 0$. The equations are set in a bounded domain $\Omega = \Omega_2 \times (-h, 0)$, where h is a positive constant and $\Omega_2 \subseteq \mathbb{R}^2$ is a smooth bounded domain.

The boundary conditions are the following. On the top we have $\partial v/\partial x_3 = 0$ and $u_3 = 0$, for $(x_1, x_2, x_3) \in \Gamma_t = \Omega_2 \times \{0\}$. while on the bottom, we assume $v = 0$ and $u_3 = 0$ for $(x_1, x_2, x_3) \in \Gamma_b = \Omega_2 \times \{-h\}$. On the side, we have $v = 0$ for $(x_1, x_2, x_3) \in \Gamma_s = \partial\Omega_2 \times [-h, 0]$. The full primitive system also contains the equations for the temperature and the salinity, and those can be added without any additional difficulties. Certain modifications are required in the case of the varying bottom, as well as in the case of physical boundary conditions on the top $\partial v/\partial x_3 + \alpha v = 0$ and $u_3 = 0$ for $(x_1, x_2, x_3) \in \Gamma_t = \Omega_2 \times \{0\}$. These modifications are given in [4]. Let

$$H = \left\{ v \in \left(L^2(\Omega)\right)^2\!: \ \mathrm{div}_2 \int_{-h}^{0} v \,\mathrm{d}x_3 = 0 \text{ on } \Omega_2, \left(\int_{-h}^{0} v \,\mathrm{d}x_3 \right) \cdot n = 0 \text{ on } \Gamma_s \right\},$$

and $V = \{v \in H \cap H^1\!: \ v = 0 \text{ on } \Gamma_b \cup \Gamma_s\}$. The norms on H and V are denoted by $\|\cdot\|_H = \|\cdot\|_{L^2}$ and $\|\cdot\|_V$ respectively. We denote by A the Stokes-type operator associated with the primitive equations; that is $Av = -P\Delta v$, where P is the L^2-orthogonal projection onto H. Let $f \in L^2_{\mathrm{loc}}([0, \infty), L^2(\Omega)^2)$. Then for all $v_0 \in H$, there exists a weak solution $v \in L^\infty_{\mathrm{loc}}([0, \infty), H) \cap L^2_{\mathrm{loc}}([0, \infty), V)$ [10], and the solution satisfies the energy inequality

$$\frac{1}{2} \int_{\Omega} |v|^2\big|_{t_1} + \nu \sum_{j=1}^{3} \sum_{k=1}^{2} \int_{t_0}^{t_1}\!\!\int_{\Omega} \partial_j v_k \partial_j v_k \leqslant \frac{1}{2} \int_{\Omega} |v|^2\big|_{t_0} + \int_{t_0}^{t_1} (v, f)_{L^2}$$

for almost every $t_0 \geqslant 0$ ($t_0 = 0$ included) and every $t_1 \geqslant t_0$. By [10,2], for every $v_0 \in V$, there exists a maximal $T_{\max} > 0$ such that there exists a strong solution $v \in L^\infty_{\mathrm{loc}}([0, T_{\max}), V) \cap L^2_{\mathrm{loc}}([0, T_{\max}), D(A))$ of the primitive equations. Also, if $T_{\max} < \infty$, then $\lim_{t \to T_{\max}} \|v(\cdot, t)\|_V = \infty$.

Theorem 2.1. *Assume that $f \in L^2_{\mathrm{loc}}([0, \infty), L^2(\Omega))$ and $v_0 \in V$. Then, there exists a unique strong solution $v \in L^\infty_{\mathrm{loc}}([0, \infty), V) \cap L^2_{\mathrm{loc}}([0, \infty), D(A))$ of the primitive equations with the initial datum u_0.*

Proof (sketch). Without loss of generality, $\nu = 1$. Assume contrary to the assertion that $T_{\max} \in (0, \infty)$, and let $T \in (0, T_{\max})$. Denote

$$\overline{E}(t) = \left(\sum_{k=1}^{2} \big\| \nabla u_k(\cdot, t) \big\|_{L^2}^2 \right)^{1/2}.$$

Choose $\delta > 0$ (which depends on $T_{\max}$) such that $\|\overline{E}\|^2_{L^2(t, t+2\delta)} \leqslant 1/\gamma$, where γ is a sufficiently large constant. Next, we find $t_j \in (j\delta, (j+1)\delta)$, where $j = 1, \ldots, l$, such that $\|\nabla v(\cdot, t_j)\|^2_{L^2} \leqslant \delta^{-1} \int_{j\delta}^{(j+1)\delta} \|\nabla v(\cdot, \tau)\|^2_{L^2} \,\mathrm{d}\tau \leqslant 1/\delta\gamma$ where l is the largest integer such that $(l+1)\delta \leqslant T$. Also, set $t_0 = 0$ and $t_{l+1} = T$. The proof consists on estimating

$$J(t) = \left(\sum_{k=1}^{2} \big\| u_k(\cdot, t) \big\|_{L^6}^6 \right)^{1/6}, \qquad K(t) = \left(\sum_{k=1}^{2} \big\| \partial_3 u_k(\cdot, t) \big\|_{L^2}^2 \right)^{1/2}$$

and

$$\bar{J}(t)=\left(\sum_{k=1}^{2}\int_{\Omega}\left|\nabla\left(u_k(\cdot,t)^3\right)\right|^2\right)^{1/6},\qquad \overline{K}(t)=\left(\sum_{k=1}^{2}\left\|\nabla\partial_3 u_k(\cdot,t)\right\|_{L^2}^2\right)^{1/2}$$

on (t_j,t_{j+1}), where $j\in\{0,\dots,l\}$ is arbitrary. In order to get an estimate for J, we multiply (PE$_k$) with u_k^5, where $k=1,2$, integrate over Ω, and add. The pressure term $-\sum_{k=1}^{2}\int u_k^5\partial_k p=-h\sum_{k=1}^{2}\int M(u_k^5)\partial_k p$ is bounded from above by $CJ^2\bar{J}^3\|\nabla_2 p\|_{L^{3/2}}$. (Note that we used independence of p on x_3.) Here, M is the vertical averaging operator $Mw(x_1,x_2)=h^{-1}\int_{-h}^{0}w(x_1,x_2,x_3)\,\mathrm{d}x_3$. We get $(\mathrm{d}/\mathrm{d}t)J^6\leqslant C\|\nabla_2 p\|_{L^{3/2}}^2J^4+CF^2J^4$, and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}J^4\leqslant C\|\nabla_2 p\|_{L^{3/2}}^2J^2+CF^2J^2,$$

where $F(t)=(\sum_{k=1}^{2}\|f_k(\cdot,t)\|_{L^2}^2)^{1/2}$. In order to obtain an estimate for K and $\overline{K}$, we multiply (PE$_k$) with $-\partial_{33}u_k$, where $k=1,2$, integrate over Ω, and add. We get:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}K^2+\overline{K}^2={}&\sum_{j,k=1}^{2}\int_{\Omega}\left(\partial_{3j}u_ju_k\partial_3u_k+\int_{\Omega}\partial_3u_ju_k\partial_{3j}u_k-2\int_{\Omega}u_j\partial_{3j}u_k\partial_3u_k\right)\\&+\sum_{k=1}^{2}\left(\int_{\Omega}\partial_k p\,\partial_{33}u_k-\int_{\Omega}f_k\partial_{33}u_k\right).\end{aligned}\tag{1}$$

The first three integrals can be bounded from above by $CJK^{1/2}\overline{K}^{3/2}$. In order to estimate the pressure term, we write:

$$\left|\sum_{k=1}^{2}\int_{\Omega_2}\partial_k p(x_1,x_2)\int_{-h}^{0}\partial_{33}u_k(x_1,x_2,x_3)\,\mathrm{d}x_3\,\mathrm{d}x_1\,\mathrm{d}x_2\right|\leqslant\sum_{k=1}^{2}\|\partial_k p\|_{L^{3/2}(\Omega_2)}\left\|\partial_3u_k(\cdot,\cdot,-h)\right\|_{L^3(\Omega_2)}$$

and use the trace theorem to estimate $\|\partial_3u_k(\cdot,\cdot,-h)\|_{L^3}\leqslant C(K+\overline{K})\leqslant C\overline{K}$. Now, we need to bound $\|\nabla_2 p\|_{L_t^2L_x^{3/2}}$. For this, we average the primitive equations in the third direction and obtain:

$$\partial_t(Mu_k)-\Delta_2Mu_k+\partial_k p=M\partial_{33}u_k-\sum_{j=1}^{2}M\partial_j(u_ju_k)+Mf_k$$

for $k=1,2$, with $\partial_1Mu_1+\partial_2Mu_2=0$. The theorem of Sohr and von Wahl [9] applied to the equation for Mv then leads to

$$\begin{aligned}\|\nabla_2 p\|_{L_t^2L_x^{3/2}}\leqslant{}&C\sum_{j=1}^{2}\left\|\partial_3u_j(\cdot,\cdot,-h,\cdot)\right\|_{L_t^2L_x^{3/2}}+C\sum_{j,k=1}^{2}\|u_j\partial_ju_k\|_{L_t^2L_x^{3/2}}\\&+C\sum_{k=1}^{2}\|f_k\|_{L_t^2L_x^2}+C\sum_{k=1}^{2}\left\|\nabla u_k(\cdot,t_j)\right\|_{L^2},\end{aligned}$$

which is less than or equal to $C\|K\|_{L_t^2}^{1/2}\|\overline{K}\|_{L_t^2}^{1/2}+C\|J\overline{E}\|_{L_t^2}+C\|F\|_{L_t^2}+C\sum_{k=1}^{2}\|\nabla u_k(\cdot,t_j)\|_{L^2}$. Using the pressure estimate, we get:

$$\begin{aligned}J(t)^4\leqslant{}&J(t_j)^4+C\|K\|_{L_t^2}\|\overline{K}\|_{L_t^2}\sup_{t_j\leqslant t\leqslant t_{j+1}}J(t)^2+C\|\overline{E}\|_{L_t^2}^2\sup_{t_j\leqslant t\leqslant t_{j+1}}J(t)^4\\&+C\left(\|F\|_{L_t^2}^2+\sum_{k=1}^{2}\left\|\nabla u_k(\cdot,t_j)\right\|_{L^2}^2\right)\sup_{t_j\leqslant t\leqslant t_{j+1}}J(t)^2,\end{aligned}\tag{2}$$

for $t\in[t_j,t_{j+1})$, while (1) leads to

$$K(t)^2 + \|\overline{K}\|^2_{L^2_t(t_j,t_{j+1})} \leqslant C\|K\|^2_{L^2_t} \sup_{t_j\leqslant t\leqslant t_{j+1}} J(t)^4 + C\|K\|_{L^2_t}\|\overline{K}\|_{L^2_t(t_j,t_{j+1})}$$
$$+ C\|\overline{E}\|^2_{L^2_t} \sup_{t_j\leqslant t\leqslant t_{j+1}} J(t)^2 + C\left(\|F\|^2_{L^2_t} + \sum_{k=1}^{2}\left\|\nabla u_k(\cdot,t_j)\right\|^2_{L^2}\right), \tag{3}$$

for $t \in [t_j, t_{j+1})$. After a short computation, we conclude that (2) and (3) imply:

$$\sup_{t_j\leqslant t\leqslant t_{j+1}} J(t)^4 + \sup_{t_j\leqslant t\leqslant t_{j+1}} K(t)^2 + \|\overline{K}\|^2_{L^2_t(t_j,t_{j+1})}$$
$$\leqslant J(t_j)^4 + \frac{C}{\gamma}\|\overline{K}\|^2_{L^2_t} + \frac{C}{\gamma}\sup_{t_j\leqslant t\leqslant t_{j+1}} J(t)^4 + C\|F\|^4_{L^2_t} + C\sum_{k=1}^{2}\left\|\nabla u_k(\cdot,t_j)\right\|^4_{L^2}$$
$$+ \frac{C}{\gamma} + \frac{C}{\gamma}\sup_{t_j\leqslant t\leqslant t_{j+1}} J(t)^2 + C\|F\|^2_{L^2_t} + C\sum_{k=1}^{2}\left\|\nabla u_k(\cdot,t_j)\right\|^2_{L^2},$$

where we used $K(t) \leqslant \overline{E}(t)$. If γ is a large enough positive constant, the sum of the second and the third term on the right-hand side can be absorbed in the half of the left-hand side. Hence, $\sup_{t_j\leqslant t\leqslant t_{j+1}} J(t)^4 + \sup_{t_j\leqslant t\leqslant t_{j+1}} K(t)^2 + \|\overline{K}\|^2_{L^2_t(t_j,t_{j+1})} \leqslant C\delta^{-2} + C + C\|F\|^4_{L^2_t}$. Then, by induction, we obtain uniform boundedness of $J(t)$, $K(t)$, and $\int_0^t \overline{K}^2$ up to T. From here, it is then not difficult to show that $\|v(\cdot,t)\|_V$ remains bounded on $(0, T_{\max})$, which contradicts $T_{\max} < \infty$. For more details, cf. [4]. □

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