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Probability Theory

A Riemann zeta stochastic process

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Abstract

It is well-known that for every $\sigma > 1$ the function $t \mapsto \zeta(\sigma + \mathrm{i}t)/\zeta(\sigma)$ represents the characteristic function of an infinitely divisible probability distribution. The purpose of this Note is to present a construction of a stochastic process having these distributions as its marginals. Functional limit theorems for this 'zeta process' and other related processes are also indicated. ***To cite this article: W. Ehm, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Un processus zeta stochastique de Riemann. Il est bien connu que pour tout $\sigma > 1$ la fonction $t \mapsto \zeta(\sigma + \mathrm{i}t)/\zeta(\sigma)$ représente la fonction caractéristique d'une loi de probabilité infiniment divisible. L'objectif de cette Note est de présenter une construction d'un processus aléatoire possédant ces lois marginales. Des théorèmes limite fonctionnels pour ce « processus zeta » et d'autres processus voisins sont indiqués également. ***Pour citer cet article : W. Ehm, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

The Riemann zeta function can be defined in the half-plane $\Re s = \sigma > 1$ by the product (over all primes) $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$. This representation underlies the observation going back to Khintchine (1938) that for every $\sigma > 1$ the function $\zeta_\sigma(t) = \zeta(\sigma + \mathrm{i}t)/\zeta(\sigma)$ is the characteristic function of an infinitely divisible probability distribution. Indeed, $\zeta_\sigma(t)$ can be written as

$$\frac{\zeta(\sigma + \mathrm{i}t)}{\zeta(\sigma)} = \prod_p \frac{1 - p^{-\sigma}}{1 - p^{-\sigma - \mathrm{i}t}} = \exp\left[\sum_p \sum_{n=1}^{\infty} \frac{p^{-\sigma n}}{n}\left(\mathrm{e}^{-\mathrm{i}tn\log p} - 1\right)\right], \tag{1}$$

and thus be represented (for $\sigma > 1$) as a product of terms of the form $\exp(a(\mathrm{e}^{\mathrm{i}bt} - 1))$, each of which is the characteristic function of a Poisson random variable with intensity a and values in the lattice kb, $k = 0, 1, 2, \ldots$. Cf. Gnedenko and Kolmogorov [6, p. 75].

Faced with a family of 'zeta distributions' indexed by parameter $\sigma > 1$, one is led to ask for *joint* distributions, i.e., for a stochastic process with time parameter σ having these distributions as its marginals. Such a 'zeta process' was

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constructed by Alexander, Baclawski and Rota [1] for discrete time indices $\sigma = 2, 3, \ldots$. Our object here is to propose an elementary construction of a continuous time zeta process. Of particular interest is, then, its limiting behavior as $\sigma \downarrow 1$.

Instead of expanding $\log(1 - p^{-\sigma-\mathrm{i}t})$ in a Taylor series as in (1), we stay with the initial product representation and note that each factor, $(1 - p^{-\sigma})/(1 - p^{-\sigma-\mathrm{i}t})$, represents the characteristic function of a random variable of the form $-Y_p(p^{-\sigma})\log p$, where $Y(u)$ denotes a geometrically distributed random variable with parameter $u \in (0, 1)$, $P[Y(u) = n] = (1 - u)u^n$ $(n = 0, 1, 2, \ldots)$. Therefore, up to a convenient change of sign, a random variable $Z(\sigma)$ with characteristic function $\zeta_\sigma(t)$ can be represented in the form

$$Z(\sigma) = \sum_p Y_p\big(p^{-\sigma}\big)\log p, \tag{2}$$

where the $Y_p(p^{-\sigma})$s are independent and geometrically distributed with parameter $p^{-\sigma}$. Our goal of constructing a *zeta process* is thus reduced to the task of constructing a *geometric process* in a natural manner. Regarding representation (2), note that the sum over the primes is almost surely a finite sum by the Borel–Cantelli lemma, since $\sum_p P\ [Y_p(p^{-\sigma}) > 0] = \sum_p p^{-\sigma} < \infty$. Thus in view of the unique prime factorization of the natural numbers we may write $Z(\sigma) = \log N(\sigma)$, where $N(\sigma) = \prod_p p^{Y_p(p^{-\sigma})}$.

Various aspects of the (one-dimensional) zeta distribution including, e.g., the representation (2) as a linear combination of geometric random variables, were discussed in [8,4,7]. For a survey of probabilistic interpretations of the zeta and other related functions and their connections with Brownian motion see Biane, Pitman and Yor [2].

2. The geometric process

Let $U_1, U_2, \ldots$ be independent and uniformly distributed on the unit interval. The *geometric process* $Y(u)$, $u \in [0, 1)$, is then defined as follows,

$$Y(u) = \begin{cases} \max\{n \geqslant 1\colon\ U_1 \leqslant u, U_2 \leqslant u, \ldots, U_n \leqslant u\} & \text{if } U_1 \leqslant u, \\ 0 & \text{if } U_1 > u. \end{cases} \tag{3}$$

We also could have set $Y(u) = \#\{n \geqslant 1\colon\ \overline{U}_n \leqslant u\}$ where $\overline{U}_n = \max\{U_k,\ k \leqslant n\}$, so that Y is the inverse of the maximum process $n \mapsto \overline{U}_n$. Evidently, $Y(u)$ is geometrically distributed with parameter u for every u, and the sample paths are non-decreasing and right continuous. Clearly also, the process is Markovian, with the following transition probabilities. Let $m \geqslant 0$ be an integer and $0 \leqslant u < v < 1$. Then

$$P\big[Y(v) = m \mid Y(u) = m\big] = \frac{P[U_1 \leqslant u, \ldots, U_m \leqslant u, U_{m+1} > v]}{P[U_1 \leqslant u, \ldots, U_m \leqslant u, U_{m+1} > u]} = \frac{u^m(1-v)}{u^m(1-u)} = \frac{1-v}{1-u},$$

and for $n \geqslant 1$

$$\begin{aligned} &P\big[Y(v) = m + n \mid Y(u) = m\big] \\ &\quad = \frac{P[U_1 \leqslant u, \ldots, U_m \leqslant u,\ u < U_{m+1} \leqslant v, U_{m+2} \leqslant v, \ldots, U_{m+n} \leqslant v, U_{m+n+1} > v]}{P[U_1 \leqslant u, \ldots, U_m \leqslant u, U_{m+1} > u]} \\ &\quad = \frac{u^m(v-u)v^{n-1}(1-v)}{u^m(1-u)} = \frac{(v-u)(1-v)v^{n-1}}{1-u}. \end{aligned}$$

These probabilities do not depend on the state m, so *Y has independent increments* that are inhomogeneous in 'time', however. The characteristic function of an increment is easily calculated as

$$E\exp\big[\mathrm{i}t\big(Y(v) - Y(u)\big)\big] = \frac{1-v}{1-v\mathrm{e}^{\mathrm{i}t}} \Big/ \frac{1-u}{1-u\mathrm{e}^{\mathrm{i}t}} =: \gamma_{u,v}(t). \tag{4}$$

Since the characteristic function of the geometric variable $Y(u)$ is $(1-u)/(1-u\mathrm{e}^{\mathrm{i}t}) = \gamma_{0,u}(t)$, (4) invites a telescoping products representation, implying that the increments are *infinitely divisible in the generalized sense that*

$$\gamma_{u_0,u_n}(t) = \prod_{k=1}^{n} \gamma_{u_{k-1},u_k}(t), \quad 0 \leqslant u_0 < u_1 < \cdots < u_n < 1. \tag{5}$$

The standard notion of infinite divisibility would require that $\gamma_{u,v}(t) = \gamma_{0,v-u}(t)$. Such homogeneity cannot be achieved by a time change. In fact, the mean and variance of the increments are given by

$$E\big(Y(v) - Y(u)\big) = \frac{v-u}{(1-u)(1-v)} = \int_u^v \frac{1}{(1-t)^2}\,\mathrm{d}t, \tag{6}$$

$$\mathrm{Var}\big(Y(v) - Y(u)\big) = \frac{(v-u)(1-uv)}{(1-u)^2(1-v)^2} = \int_u^v \frac{1+t}{(1-t)^3}\,\mathrm{d}t, \tag{7}$$

respectively, for $0 \leqslant u \leqslant v < 1$. The non-linearity of the variance function, $\mathrm{Var}\, Y(v) = v/(1-v)^2$, can be transformed away: the substitution $v \equiv v(t) = (1+2t-(1+4t)^{1/2})/(2t)$ gives $\mathrm{Var}\, Y(v(t)) = t$ $(t \geqslant 0)$. However, the expectations $EY(v(t)) = v(t)/(1-v(t)) = t(1-v(t))$ remain non-linear in t, showing that the time homogeneity of the increments required for infinite divisibility in the usual sense cannot be achieved.

3. The zeta process, and related functional limit theorems

Let Y_p $(p = 2, 3, 5, \ldots)$ be independent geometric processes. Then the (*Riemann*) *zeta* (*stochastic*) *process* $(1,\infty) \ni \sigma \mapsto Z(\sigma)$ is defined by (2), with the understanding that the arguments of the Y_ps now represent (transformed) 'time' in addition to their rôle as parameter of a geometric distribution. Let us state its basic properties.

The zeta process inherits the independent increments, or, since transformed times $p^{-\sigma}$ run backwards, *decrements* property from its component processes. The expectations are given by

$$EZ(\sigma) = \sum_p (\log p) p^{-\sigma}/\big(1 - p^{-\sigma}\big) = -(\log\zeta)'(\sigma).$$

Therefore, the process

$$\sigma \mapsto R(\sigma) = Z(\sigma) + (\log\zeta)'(\sigma) = \sum_p \log p\big(Y_p\big(p^{-\sigma}\big) - p^{-\sigma}/\big(1-p^{-\sigma}\big)\big)$$

is a backward martingale w.r.t. the filtration $\mathcal{G} = \{\mathcal{G}(\sigma), \sigma > 1\}$, where $\mathcal{G}(\sigma) = \bigvee_p \mathcal{F}_p(p^{-\sigma})$ and $\mathcal{F}_p$ denotes the filtration associated with Y_p, $\mathcal{F}_p(v) = \sigma\{Y_p(u), 0 \leqslant u \leqslant v\}$. By independence across ps the variance of $R(\sigma)$ (or $Z(\sigma)$) is

$$V(\sigma) := \mathrm{Var}\, R(\sigma) = \sum_p (\log p)^2 p^{-\sigma}/\big(1-p^{-\sigma}\big)^2 = (\log\zeta)''(\sigma).$$

The pole of the zeta function at $\sigma = 1$ implies that $EZ(\sigma)$ and $\mathrm{Var}\, Z(\sigma)$ both diverge to infinity as $\sigma \downarrow 1$.

To state the functional limit theorem for the zeta process it is convenient to reverse the 'time' direction to 'forward' by passing to new time parameters $u \in [0,1]$ as follows. The function $\sigma \mapsto V(\sigma)$ decreases strictly and continuously on $(1,\infty)$, from $V(1+) = \infty$ to $V(\infty) = 0$. Therefore, given any $T > 0$ there exists for every $u \in (0,1]$ a uniquely defined $\sigma_T(u) > 1$ such that $V(\sigma_T(u)) = uT$. In particular, $V(\sigma_T(1)) = T$, and it makes sense to define $\sigma_T(0) = \infty$, so that $\sigma_T(u)$ decreases from ∞ to $\sigma_T(1)$ as u runs from 0 to 1. Moreover, $\sigma_T(1) \downarrow 1$ as $T \uparrow \infty$ (and vice versa). Then, given any $T > 0$, let process η_T be defined by

$$\eta_T(u) = \sqrt{u} + T^{-1/2} R\big(\sigma_T(u)\big) \quad (0 \leqslant u \leqslant 1).$$

Weak convergence of processes is understood in the space D of càdlàg functions on [0, 1] endowed with the Skorokhod topology.

Theorem 3.1. *As $T \to \infty$, the process $\eta_T = \{\eta_T(u), 0 \leqslant u \leqslant 1\}$ converges weakly to a process $\eta = \{\eta(u), 0 \leqslant u \leqslant 1\}$ characterized by the following properties*: (i) *η has independent increments*; (ii) *for $u \in [0,1]$, $\eta(u)$ is exponentially distributed with expectation $\sqrt{u}$, $\eta(u) \sim \mathcal{E}(\sqrt{u})$. In particular, for $u = 1$ it follows that*

$$(\sigma - 1)\log N(\sigma) \longrightarrow_d \mathcal{E}(1) \quad \text{as } \sigma \downarrow 1. \tag{8}$$

The nature of the 'scaled exponentials' process η becomes clear on noting that for any $0 \leqslant u < v \leqslant 1$ the distribution of the increment $\eta(v) - \eta(u)$ is a mixture of the point mass at zero and the distribution $\mathcal{E}(\sqrt{v})$, with weights $\sqrt{u/v}$ and $1 - \sqrt{u/v}$, respectively. In fact, independence of the increments of η implies, similarly as at (5), a telescoping products representation of the characteristic function $(1 - \mathrm{i}t\sqrt{v})^{-1}$ of $\mathcal{E}(\sqrt{v})$. In particular, $E\exp[\mathrm{i}t(\eta(v) - \eta(u))] = (1 - \mathrm{i}t\sqrt{u})/(1 - \mathrm{i}t\sqrt{v})$, which quotient is readily identified as the characteristic function of the above mixture distribution. The limit process thus mimics the structure of the geometric process.

Concerning the *proof of Theorem 3.1* let us indicate the argument for the convergence in distribution of $\eta_T(1)$ to $\mathcal{E}(1)$. The limit distributions of $\eta_T(u)$ for $u < 1$ then follow by a scaling argument. Tightness can be proved using a suitable fluctuation inequality along with the independence of the increments. To ease the notation let us write $\sigma_T(1) = \sigma$. Then, given $T = V(\sigma) > 0$ and $b \in \mathbf{R}$ let

$$K_\sigma(b) = T^{-1} \sum_p (\log p)^2 E X_p(\sigma)^2 \chi\big(X_p(\sigma) \leqslant bT^{1/2}/\log p\big)$$

where the random variables $X_p(\sigma) = Y_p(p^{-\sigma}) - p^{-\sigma}/(1 - p^{-\sigma})$ are independent across primes p, and $\chi(\cdot)$ denotes the indicator function of the event in brackets. By a theorem of Brown and Eagleson [3] it suffices to show that

$$\lim_{\sigma \downarrow 1} K_\sigma(b) = \big(1 - \mathrm{e}^{-b}(1 + b)\big)_+ =: K(b) \tag{9}$$

for every $b \in \mathbf{R}$ and to note that

$$\int_0^\infty \big(\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx\big)x^{-2}\,\mathrm{d}K(x) = \int_0^\infty \big(\mathrm{e}^{\mathrm{i}tx} - 1\big)x^{-1}\mathrm{e}^{-x}\,\mathrm{d}x - \mathrm{i}t = \log\frac{\mathrm{e}^{-\mathrm{i}t}}{1 - \mathrm{i}t}$$

is the log characteristic function of a standard exponentially distributed random variable minus 1. For the proof of (9) one writes

$$K_\sigma(b) = \sum_{n=0}^\infty \Bigg[\sum_p \frac{(\log p)^2}{T}\frac{1 - p^{-\sigma}}{p^{\sigma n}}\bigg(n - \frac{p^{-\sigma}}{1 - p^{-\sigma}}\bigg)^2 \chi\bigg(n - \frac{p^{-\sigma}}{1 - p^{-\sigma}} \leqslant \frac{bT^{1/2}}{\log p}\bigg)\Bigg] \equiv \sum_{n=0}^\infty A_n(\sigma, b)$$

and shows that as $\sigma \downarrow 1$ one has (i) $\sum_{n \neq 1} A_n(\sigma, b) \to 0$ and (ii) $A_1(\sigma, b) \to K(b)$, using the prime number theorem for the second assertion.

Let us conclude by stating a related result that may be regarded as a strictly stochastic, functional version of the classical Erdős–Kac theorem [5]. The number of prime factors of $N(\sigma) = \prod_p p^{Y_p(p^{-\sigma})}$ counting multiplicities is $\sum_p Y_p(p^{-\sigma})$. Similarly as above, $M(\sigma) = \sum_p (Y_p(p^{-\sigma}) - \frac{p^{-\sigma}}{1 - p^{-\sigma}})$, $\sigma > 1$, is a backward martingale (w.r.t. the filtration $\mathcal{G}$) with independent decrements and variance function $V(\sigma) = \sum_p p^{-\sigma}/(1 - p^{-\sigma})^2$. Given $T > 0$ and $u \in [0, 1]$ let $\sigma_T(u) > 1$ be defined by the relation $V(\sigma_T(u)) = uT$, as above.

Theorem 3.2. *Given $T > 0$, let $W_T(u) = T^{-1/2}M(\sigma_T(u))$ $(0 \leqslant u \leqslant 1)$. Then as $T \to \infty$, the process $W_T = \{W_T(u),\ 0 \leqslant u \leqslant 1\}$ converges weakly to standard Brownian motion.*

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