Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 345 (2007) 307–312
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Group Theory/Algebraic Geometry

Purity of G_2-torsors

Vladimir Chernousov [a,1], Ivan Panin [b,c,2]

[a] *Department of Mathematics, University of Alberta, Edmonton, Alberta T6G 2G1, Canada*
[b] *SFB-701 at Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany*
[c] *Steklov Mathematical Institute at St. Petersburg, Fontanka 27, 191023 St. Petersburg, Russia*

Received 18 June 2007; accepted 26 July 2007

Available online 4 September 2007

Presented by Jean-Pierre Serre

Abstract

Let k be a field of characteristic zero, and let G be a split simple algebraic group of type G_2 over k. We prove that the functor $R \mapsto H^1_{\text{ét}}(R, G)$ of G-torsors satisfies purity for regular local rings containing k. ***To cite this article: V. Chernousov, I. Panin, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Un théorème de pureté pour les G_2-torseurs. Soit k un corps de caractéristique 0, et soit G un k-groupe simple déployé de type G_2. Nous montrons que le foncteur des G-torseurs satisfait au « théorème de pureté » pour la catégorie des anneaux locaux réguliers contenant k. ***Pour citer cet article : V. Chernousov, I. Panin, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

Soient $\mathcal{F}$ un foncteur covariant de la catégorie des anneaux commutatifs vers celle des ensembles et R un anneau intègre de corps des fractions K. On dit qu'un élément $\xi \in \mathcal{F}(K)$ est *non ramifié* en l'idéal premier $\mathfrak{P} \subset R$ de hauteur 1 si

$$\xi \in \operatorname{Im}\big[\mathcal{F}(R_{\mathfrak{P}}) \to \mathcal{F}(K)\big].$$

On dit que ξ est *non ramifié* s'il est non ramifié en tout idéal premier de hauteur 1 de R. L'ensemble de tous les éléments non ramifiés de $\mathcal{F}(K)$ est noté $\mathcal{F}(K)_{\text{ur}}$. On dit que le foncteur $\mathcal{F}$ *satisfait à la condition de pureté pour un anneau intègre* R si

$$\operatorname{Im}\big[\mathcal{F}(R) \to \mathcal{F}(K)\big] = \mathcal{F}(K)_{\text{ur}}.$$

E-mail address: chernous@math.ualberta.ca (V. Chernousov).
[1] Supported by the Canada Research Chairs Program, and by NSERC research grant.
[2] Supported by the Presidium of RAS Program "Fundamental Research", the RFFI-grant, INTAS-03-51-3251 and INTAS-05-1000008-8118.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2007.07.018

Dans cette note, nous considérons les deux foncteurs suivants :

$$\mathcal{T}(R) = H^1_{\text{ét}}(R, G) = \{\text{classes d'isomorphisme des } G\text{-torseurs sur } R\}$$

où G est un groupe déployé de type G_2, et

$$\mathcal{P}f_3(R) = \{\text{classes d'isomorphisme de 3-formes de Pfister sur } R\}.$$

Nous démontrons que ces deux foncteurs satisfont à la condition de pureté pour les anneaux locaux réguliers contenant un corps de caractéristique nulle. Cela répond affirmativement à une question posée dans [2, Question 6.4, p. 124] sur la pureté du foncteur des G-torseurs où G est un groupe déployé de type G_2.

L'hypothèse restrictive sur la caractéristique provient du fait que nous utilisons le résultat principal de [8] sur les espaces quadratiques rationnellement isotropes, qui n'a été prouvé qu'en caractéristique nulle (la preuve utilise la résolution des singularités).

Remarquons également que jusqu'à récemment, on ne trouvait aucun résultat dans la littérature sur la pureté du foncteur des G-torseurs pour les groupes de type exceptionnel. Le théorème de pureté est connu pour certains groupes de type classique : les groupes déployés de type A_n (non publié) ; les groupes de la forme $SL_{1,A}$ où A est une algèbre simple centrale sur un corps [1] ; les groupes déployés de type B_n [8] ; les groupes simplement connexes déployés de type C_n (de manière évidente) ; certains groupes déployés de type D_n (comme le groupe spécial orthogonal d'une forme quadratique) [8].

La preuve de la pureté du foncteur des G_2-torseurs consiste à montrer d'abord la pureté pour le foncteur $\mathcal{P}f_3$ puis que le morphisme naturel $\mathcal{P}f_3(K)_{\text{ur}} \to \mathcal{T}(K)_{\text{ur}}$ est surjectif.

1. Main result

In this note we give an affirmative answer to a question raised in [2, Question 6.4, p. 124] about the purity of the functor of G-torsors in the case where G is a split group of type G_2.

Let us first recall what the purity property for a functor is. Let $\mathcal{F}$ be a covariant functor from the category of commutative rings to the category of sets, and let R be a domain with field of fractions K. We say that an element $\xi \in \mathcal{F}(K)$ is *unramified* at a prime ideal $\mathfrak{P} \subset R$ of height 1 if

$$\xi \in \operatorname{Im}\bigl[\mathcal{F}(R_{\mathfrak{P}}) \to \mathcal{F}(K)\bigr].$$

We say that ξ is *unramified* if it is unramified with respect to all prime ideals in R of height 1. It is clear that

$$\operatorname{Im}\bigl[\mathcal{F}(R) \to \mathcal{F}(K)\bigr] \subseteq \mathcal{F}(K)_{\text{ur}}$$

where $\mathcal{F}(K)_{\text{ur}}$ is the set of all unramified elements. We say that the functor $\mathcal{F}$ *satisfies purity for a domain* R if every $\xi \in \mathcal{F}(K)_{\text{ur}}$ is in the image of $\mathcal{F}(R)$, i.e. if

$$\bigcap_{\text{ht}\,\mathfrak{P}=1} \operatorname{Im}\bigl[\mathcal{F}(R_{\mathfrak{P}}) \to \mathcal{F}(K)\bigr] = \operatorname{Im}\bigl[\mathcal{F}(R) \to \mathcal{F}(K)\bigr].$$

If $\mathcal{F}$ and $\mathcal{F}'$ are two functors and $f : \mathcal{F} \to \mathcal{F}'$ is a natural transformation then

$$f\bigl(\mathcal{F}(K)_{\text{ur}}\bigr) \subseteq \mathcal{F}'(K)_{\text{ur}}.$$

In what follows we assume that 2 is invertible in R. We say that a quadratic space over R is a 3-Pfister space if the corresponding quadratic form is isomorphic to a form

$$\langle\langle a, b, c\rangle\rangle = \langle 1, -a\rangle \otimes \langle 1, -b\rangle \otimes \langle 1, -c\rangle$$

where a, b, c are units in R. We will consider the following two functors:

$$\mathcal{T}(R) = H^1_{\text{ét}}(R, G) = \{\text{isomorphism classes of } G\text{-torsors over } R\}$$

where G is a split group of type G_2, and

$$\mathcal{P}f_3(R) = \{\text{isomorphism classes of 3-fold Pfister spaces over } R\}.$$

The main results of this note are the following purity theorems:

Theorem 1. *Let G be a simple split algebraic group of type G_2 over the field* $\mathbf{Q}$ *of rational numbers. Then the functor* $\mathcal{T}: R \mapsto H^1_{\text{ét}}(R, G)$ *satisfies purity for regular local rings containing* $\mathbf{Q}$.

Theorem 2. *The functor $R \mapsto \mathcal{P}f_3(R)$ satisfies purity for regular local rings containing* $\mathbf{Q}$.

Remark 3. Until recently there were no results in the literature on the purity of the functor of G-torsors for groups of exceptional type. For certain groups of classical type the purity theorem is known; more precisely it is known for split groups of type A_n (unpublished); groups of the form $SL_{1,A}$, where A is a central simple algebra over a field [1]; split groups of type B_n [8]; split simply connected groups of type C_n (obvious); certain split groups of type D_n (like the special orthogonal group of a quadratic form) [8].

Remark 4. The characteristic restriction in the theorem is due to the fact that we use the main result in [8] on rationally isotropic quadratic spaces which was proven in characteristic zero only (the resolution of singularities is involved in that proof).

2. Quadratic spaces splitting over étale quadratic extensions

For the definition and basic properties of quadratic spaces over a commutative ring we refer to [3]. To prove purity for 3-fold Pfister forms we need information about quadratic spaces over R splitting over an étale quadratic extension of R. In this section R denotes a regular local ring. Note that R is a domain; we denote by k its residue field and by K its field of fractions.

Let $S = R[t]/(t^2 - u)$ where $u \in R^\times$ is a unit. Let $L = S \otimes_R K$ and $l = S \otimes_R k$. It is clear that L/K and l/k are étale quadratic extensions of K and k respectively. Thus L/K is either isomorphic to $K \times K$ or it is a Galois field extension of degree 2. Let $\sigma \in Gal(S/R)$ be the nontrivial automorphism. It induces the involutions of L/K and l/k which we denote for simplicity by the same letter σ.

If (V, q) is a quadratic space over R we set

$$V_S = (V_S, q_S) = (V, q) \otimes_R S, \qquad V_L = (V_L, q_L) = (V_S, q_S) \otimes_S L,$$
$$V_K = (V_K, q_K) = (V, q) \otimes_R K, \qquad V_k = (V_k, q_k) = (V, q) \otimes_R k,$$
$$V_l = (V_l, q_l) = (V_S, q_S) \otimes_R k.$$

If $w \in V_S$ and $v \in V$ we set

$$\bar{w} = w \otimes 1 \in V_l, \qquad \bar{v} = v \otimes 1 \in V_k.$$

Lemma 5. *Let (W, q) be a quadratic space over R of rank 2 such that W_S is a hyperbolic plane $\mathbb{H}_S$. Then either W is a hyperbolic plane or there exists a unit $\lambda \in R^\times$ such that $q \cong \lambda \cdot \langle 1, -u \rangle$.*

Proof. Since R is local, we can write q in a diagonal form $q = \lambda \cdot \langle 1, -d \rangle$ where $\lambda, d \in R^\times$. Assume first that q_K is hyperbolic. Then there is $f \in K$ such that $f^2 = d$. Since R is a unique factorization domain ([4, Theorem 48, page 142]) and d is a unit, we have $f \in R^\times$. Thus q splits over R.

Let now q_K be anisotropic over K. Note that L/K is a field extension since otherwise $L = K \times K$ and hence $q_L = q_K \times q_K$ would be L-anisotropic. Since q_K splits over $L = K(\sqrt{u})$ the discriminant of q_K is $u(K^\times)^2$. It follows that $df^2 = u$ for some $f \in K^\times$ or equivalently $f^2 = d^{-1}u \in R^\times$. As above, this implies $f \in R^\times$, hence $q \cong \lambda \cdot \langle 1, -u \rangle$. □

Proposition 6. *Let (V, q) be a quadratic space over R. Assume that the space (V_S, q_S) is hyperbolic. Then (V, q) can be decomposed as*

$$(V, q) = (W_1, q_1) \perp \cdots \perp (W_n, q_n)$$

where (W_i, q_i), $i = 1, \ldots, n$, are quadratic spaces of rank 2 splitting over S.

Proof. We argue by induction on n where $\dim V = 2n$. If $\dim V = 2$ there is nothing to prove. Assume $\dim V \geqslant 3$. Take a unimodular vector $v \in V_S$ with $q_S(v) = 0$. Since V_l is a hyperbolic space of rank at least 3 there is a vector $\bar{w} \in V_l$ such that

$$q_l(\bar{w}) = 0, \quad \big(\bar{w}, \sigma(\bar{w})\big) \in l^\times, \quad (\bar{w}, \bar{v}) \in l^\times$$

where $(-,-)$ is the bilinear form associated with q. It is clear that $\bar{w}$ and $\sigma(\bar{w})$ generate a hyperbolic plane over l which is a direct summand of V_l.

Take now any lifting w of $\bar{w}$. Replacing w by

$$w - \frac{(w,w)}{2(w,v)} \cdot v$$

we may additionally assume that $q_S(w) = 0$. Set $W = wS + \sigma(w)S \subseteq V_S$. Since $(w, \sigma(w)) \in S^\times$, the space W is a σ-invariant hyperbolic plane, and it is clear that $(W, q_S|_W)$ is a direct summand of (V_S, q_S).

Thus we have orthogonal quadratic space decompositions $V_S = W \oplus W^\perp$ and $V = W^\sigma \oplus (W^\perp)^\sigma$, where $W^\perp$ is the orthogonal complement of W and W^σ, $(W^\perp)^\sigma$ are σ-invariant subspaces. We also have $W^\sigma \otimes_R S = W$ and $(W^\perp)^\sigma \otimes_R S = W^\perp$ since $V \otimes_R S = V_S$. The first equality implies that W^σ splits over S. Since S is semi-local and V_S is hyperbolic, so is $W^\perp$. Applying the induction hypothesis to $(W^\perp)^\sigma$ completes the proof. □

Proposition 6 and Lemma 5 imply the following:

Corollary 7. *Let (V,q) be a quadratic space over R such that (V_S, q_S) is hyperbolic. If (V_K, q_K) is anisotropic, then there exist units $\lambda_1, \lambda_2, \ldots, \lambda_n \in R^\times$ and a quadratic space isomorphism $q \cong \langle \lambda_1, \lambda_2, \ldots, \lambda_n \rangle \otimes \langle 1, -u \rangle$.*

3. Purity of 3-fold Pfister forms

In this section we keep the above notation. In addition, we assume $\mathbb{Q} \subset R$.

Theorem 8. *Let ϕ be a 3-fold Pfister form over K. Assume that ϕ is unramified over K viewed as an 8-dimensional quadratic form. Then there exists a Pfister space $\phi' := \langle\langle a', b', c' \rangle\rangle$ over R, where $a', b', c' \in R^\times$, such that $\phi' \otimes_R K \cong \phi$.*

Proof. Reasoning first as in [6, Proof of Theorem A] and then as in [6, Proof of Theorem 7.1] we may reduce the general case to the case of a regular local ring R of the form $\mathcal{O}_{X,x}$, where X is a smooth affine variety over a field k of characteristic zero and $x \in X$ is a closed point.

If ϕ is isotropic, then it is hyperbolic and there is nothing to prove. So we may assume that ϕ is anisotropic.

By [8, Cor. 1.0.5], there exist a quadratic space ψ over R such that ψ_K is isomorphic to ϕ. Since ψ_K represents 1 over K there is a decomposition $\psi \cong \langle 1 \rangle \perp \psi'$ over R, by [8, Cor. 1.0.6]. Take a diagonalization $\psi' = \langle u_2, u_3, \ldots, u_8 \rangle$ of ψ' over R and set $u := -u_2$.

Let S and L be the étale quadratic extensions of R and K respectively corresponding to u. Since ψ is isotropic over S, hence over L, so is ϕ. It then follows that $\psi_K = \phi$ is hyperbolic over L. We claim that the space $\psi \otimes_R S$ is hyperbolic. Indeed, the ring S is regular semi-local of the form $\mathcal{O}_{Y,y}$, where Y is a k-smooth affine variety and $y \subset Y$ is a subset consisting of one or two closed points. If S is local, i.e. y is a closed point, the hyperbolicity of $\psi \otimes_R S$ was proven by Ojanguren [5]. This result was reproved by Ojanguren, Panin in [7, Proof of Theorem 5.1] and we note that the arguments therein work equally well for semi-local regular rings of a finite family of closed points on a k-smooth variety.

Since ψ_K is anisotropic, by Corollary 7 we can write ψ in the form

$$\psi = \langle 1, -u \rangle \otimes \langle b_1, b_2, b_3, b_4 \rangle = b_1\psi_1 \perp b_3\psi_2$$

where $b_1, b_2, b_3, b_4 \in R^\times$ and

$$\psi_1 = \langle 1, -u \rangle \otimes \langle 1, b_1b_2 \rangle, \qquad \psi_2 = \langle 1, -u \rangle \otimes \langle 1, b_3b_4 \rangle.$$

We claim that $\psi_1 \cong \psi_2$. Indeed, by a theorem of Ojanguren [5] it suffices to check this over K. The function field $K(\psi_1)$ of the quadric $\psi_{1,K} = 0$ splits both $\psi_{1,K}$ and ψ_K. By Witt cancellation it splits $\psi_{2,K}$ as well. It follows that

ψ_1 and ψ_2 are K-isomorphic, since they are 2-fold Pfister forms. Thus ψ can be written in the form $\psi = b_1\eta$ where η is the 3-fold Pfister form

$$\eta = \langle 1, b_1 b_3\rangle \otimes \psi_1.$$

It remains to show that $b_1\eta$ is isomorphic to η over R or equivalently, by a theorem of Ojanguren [5], over K. In turn, this is equivalent to the property that η_K represents b_1. But this is clear, since $\psi_K = b_1\eta$ and ψ represents 1. □

Corollary 9. *The functor $\mathcal{P}f_3$ satisfies purity for regular local rings containing $\mathbb{Q}$.*

4. Proof of Theorem 1

It is known that a split R-group G of type G_2 is the automorphism group of a split octonion algebra $\mathbf{O}$ and this gives rise to an embedding $i : G \hookrightarrow \mathrm{O}_8$ of G into a split orthogonal group O_8 in dimension 8. Let $1, e_1, e_2, \dots, e_7$ be a canonical basis of $\mathbf{O}$ with

$$e_1^2 = 1, \quad e_2^2 = 1, \quad e_3^2 = 1,$$
$$e_4 = e_1e_2 = -e_2e_1, \quad e_5 = e_2e_3 = -e_3e_2,$$
$$e_6 = e_3e_4 = -e_4e_3, \quad e_7 = e_4e_5 = -e_5e_4.$$

It follows from these relations that a mapping

$$e_1 \to \epsilon_1 e_1, \quad e_2 \to \epsilon_2 e_2, \quad e_3 \to \epsilon_3 e_3$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$ can be extended to an automorphism of $\mathbf{O}$ and this gives rise to an embedding $j : \mu_2 \times \mu_2 \times \mu_2 \hookrightarrow G$.

Consider the following diagram of pointed sets

$$\begin{array}{ccccc} H^1_{\text{ét}}(K, \mu_2 \times \mu_2 \times \mu_2) & \xrightarrow{j_K} & H^1_{\text{ét}}(K, G) & \xrightarrow{i_K} & H^1_{\text{ét}}(K, \mathrm{O}_8) \\ \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle\gamma} \\ H^1_{\text{ét}}(R, \mu_2 \times \mu_2 \times \mu_2) & \xrightarrow{j_R} & H^1_{\text{ét}}(R, G) & \xrightarrow{i_R} & H^1_{\text{ét}}(R, \mathrm{O}_8). \end{array}$$

Recall that the set $H^1_{\text{ét}}(R, \mathrm{O}_8)$ (resp. $H^1_{\text{ét}}(K, \mathrm{O}_8)$) classifies isomorphism classes of quadratic forms of dimension 8 over R (resp. over K). It is known [9, Th. 1.7.1, page 16] that i_K is injective and the image of i_K coincides with the set of isomorphism classes of 3-fold Pfister forms. Since R is local, we have $H^1_{\text{ét}}(R, \mu_2) \simeq R^\times / R^{\times 2}$. If a, b, c are units in R (resp. in K), we write (a, b, c) for the corresponding element in $H^1_{\text{ét}}(R, \mu_2 \times \mu_2 \times \mu_2)$ (resp. in $H^1_{\text{ét}}(K, \mu_2 \times \mu_2 \times \mu_2)$). It follows from our construction of j that $(i_R \circ j_R)(a, b, c)$ (resp. $(i_K \circ j_K)(a, b, c)$) is the Pfister space $\langle\langle a, b, c\rangle\rangle$ over R (resp. K).

Let $\xi \in H^1_{\text{ét}}(K, G)$ be an unramified class, and let ϕ be a quadratic form over K representing the class $i_K(\xi)$. We mentioned above that ϕ is a 3-fold Pfister form. Since

$$i_K\big(H^1(K, G)_{\text{ur}}\big) \subset H^1(K, O_8)_{\text{ur}},$$

the class $[\phi]$ is unramified. By Theorem 8 we have a K-quadratic space isomorphism $\phi \simeq \langle\langle a, b, c\rangle\rangle$ for some $a, b, c \in R^\times$. Set $\tilde{\xi} = j_R(a, b, c)$. We claim that $\tilde{\xi}_K = \beta(\tilde{\xi}) = \xi$. Indeed,

$$i_K(\tilde{\xi}_K) = \big(i_R\big(j_R(a, b, c)\big)\big)_K = (\gamma \circ i_R \circ j_R)(a, b, c) = \phi = i_K(\xi).$$

Since i_K is injective, $\tilde{\xi}_K = \xi$. This completes the proof.

Remark 10. J.-P. Serre pointed out to us that if R is any local ring in which 2 is invertible then the following three categories are equivalent:

(i) *the category of G_2-torsors over R*;
(ii) *the category of isomorphism classes of octonion algebras over R*;
(iii) *the category of isomorphism classes of* 3*-fold Pfister spaces over R.*

This shows that Theorem 1 is a straightforward consequence of Corollary 9.

Acknowledgements

We are grateful to J.-P. Serre for helpful comments.

References

[1] J.-L. Colliot-Thélène, M. Ojanguren, Espaces principaux homogènes localement triviaux, Publ. Math. IHES 75 (2) (1992) 97–122.
[2] J.-L. Colliot-Thélène, J.-J. Sansuc, Fibrés quadratiques et composantes connexes réelles, Math. Annalen 244 (1979) 105–134.
[3] M.-A. Knus, Quadratic and Hermitian Forms over Rings, Grundlehren der Math. Wissenschaften, vol. 294, Springer, 1991.
[4] H. Matsumura, Commutative Algebra, W.A. Benjamin Co., New York, 1970.
[5] M. Ojanguren, Quadratic forms over regular rings, J. Indian Math. Soc. 44 (1980) 109–116.
[6] M. Ojanguren, I. Panin, A purity theorem for the Witt group, Ann. Sci. Ecole Norm. Sup. (4) 32 (1) (1999) 71–86.
[7] M. Ojanguren, I. Panin, Rationally trivial Hermitian spaces are locally trivial, Math. Z. 237 (2001) 181–198.
[8] I. Panin, Rationally isotropic quadratic spaces are locally isotropic, http://www.math.uiuc.edu/K-theory/0671/2003.
[9] T.A. Springer, F.D. Veldkamp, Octonions, Jordan Algebras and Exceptional Groups, Springer-Verlag, 2000.