Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 345 (2007) 381–384
COMPTES RENDUS MATHEMATIQUE
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Partial Differential Equations

High frequency periodic solutions of semilinear equations

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Received 18 March 2007; accepted after revision 18 July 2007

Available online 20 September 2007

Presented by Haïm Brezis

Abstract

We are interested with positive solutions of $-\varepsilon^2\Delta u + f(u) = 0$ in $S^1 \times \mathbb{R}$, i.e. periodic solutions in the first coordinate x_1. The model function for f is $f(u) = u - u^p$, $p > 1$. We prove that for ε large enough, any positive solution is a function of the second coordinate only. ***To cite this article: G. Allain, A. Beaulieu, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Solutions périodiques de haute fréquence d'équations semi-linéaires. On s'intéresse aux solutions positives de $-\varepsilon^2\Delta u + f(u) = 0$ dans $S^1 \times \mathbb{R}$, c'est-à-dire aux solutions périodiques en x_1, la première coordonnée. Le cas modèle est $f(u) = u - u^p$, $p > 1$. Nous prouvons que, pour ε suffisamment grand, toute solution positive est une fonction de x_2 seulement. ***Pour citer cet article : G. Allain, A. Beaulieu, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

Let $N \geqslant 2$. Under some conditions on f, following Kwong and Zhang, [9], there exists a ground-state solution w_0, that is a radial positive solution, of

$$-\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^{N-1}. \tag{1}$$

Dancer, [5], studied the bifurcation of solutions, which are periodic in one variable, of

$$-\varepsilon^2\Delta u + f(u) = 0 \quad \text{in } S^1 \times \mathbb{R}^{N-1} \tag{2}$$

around $w_\varepsilon(x_1, x') = w_0(\frac{x'}{\varepsilon})$, which is seen as a bounded solution in $\mathbb{R}^N$, depending only on $N-1$ variables. There exists a sequence (ε_j) of positive parameters, with $\varepsilon_j = \varepsilon_0/(j+1)$ for $j \in \mathbb{N}$, such that there is a curve of positive solutions of (2) in $L^\infty(\mathbb{R}^N)$ which are 2π-periodic in x_1, and decay to zero, uniformly in x_1, as $|x'| \to \infty$ and which

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2007.07.010

bifurcate from w_{ε_j}. We could ask whether w_ε is the only positive bounded periodic solution of (2) for $\varepsilon > \varepsilon_0$. In all what follows we suppose that $N = 2$ and we give a partial answer to this question in this case.

The model function for f is $f(u) = u - u^p$, $p > 1$, but we give more general assumptions for a continuous function f:

$$\text{There exists } s_0 > 0 \text{ such that } f \text{ is non-decreasing in } [0, s_0]. \tag{3}$$

$$f(0) = 0 \text{ and } f'(0) \text{ exists.} \tag{4}$$

$$\text{There exists } p > 1 \text{ and } K > 0 \text{ such that for any } u > 0, -Ku^p \leqslant f(u) - f'(0)u < 0. \tag{5}$$

Theorem 1.1. *Let f be a C^1 function in $\mathbb{R}^+$, that satisfies the hypotheses* (3), (4) *and* (5)*, such that f' is decreasing in $\mathbb{R}^+$, f has a maximum for some $c > 0$ and f'' exists and is continuous, except in isolated points of $\mathbb{R}^+$. Then there exists $\bar{\varepsilon} > 0$ such that for $\varepsilon > \bar{\varepsilon}$ any positive solution of* (2) *that tends to* 0 *as $|x_2|$ tends to infinity, uniformly in $x_1 \in S^1$, can only be a function of the variable x_2.*

Therefore, when $f(u) = u - u^p$, $p > 1$, for $\varepsilon > \bar{\varepsilon}$, the solutions are the null solution and the functions $w_0(\frac{x_2}{\varepsilon})$, and the functions obtained by translation from these. Since the conjecture of De Giorgi, (see [1]), several authors ([6,1,3], ...) have proved that the solution of some other semilinear elliptic equations on $\mathbb{R}^N$ depends only on one variable.

2. Some properties of solutions

Theorem 2.1. *Let f be a function that verifies* (3) *and* (4)*. Let $(x_1, x_2) \mapsto u(x_1, x_2)$ be a positive solution of* (2) *that tends to* 0 *as x_2 tends to infinity, uniformly in $x_1 \in S^1$. Then there exists $t_0 \in \mathbb{R}$ such that $u(x_1, t_0 - x_2) = u(x_1, t_0 + x_2)$ for all $(x_1, x_2) \in S^1 \times \mathbb{R}$ and u decreases with respect to x_2 for $x_2 \geqslant t_0$.*

The proof of this theorem is similar to [2]. It uses the moving plane method like [7,4].

Theorem 2.2. *Let f be a function that verifies* (3), (4) *and* (5)*. Then for all $\varepsilon > 0$, there exists $C > 0$, depending only on ε, $f'(0)$ and p, and decreasing with respect ε, such that if u is any positive solution of* (2) *that satisfies the hypotheses of Theorem* 1.1 *and that is even in x_2, we have*

$$\sup_{S^1 \times \mathbb{R}^+} u \leqslant C\left(\inf_{S^1 \times \{0\}} u + \frac{K}{\varepsilon^2} \inf_{S^1 \times \{0\}} u^p\right). \tag{6}$$

Proof. The claim follows from the Harnack inequalities. First, we apply Theorem 8.17 of [8] with $Lu = \Delta u$ and the equation $Lu = \frac{1}{\varepsilon^2} f(u)$ and $R = \pi$. We get for all $n > 1$ and all $q > 2$ a constant C that depends on n and q, such that for all positive solution u and all $\varepsilon > 0$ we have

$$\sup_{B_R(0)} u \leqslant C\left(R^{-\frac{2}{n}} \|u\|_{L^n(B_{2R}(0))} + \frac{1}{\varepsilon^2} R^{2-\frac{4}{q}} \left(\int_{B_{2R}(0)} \left(f'(0)u + Ku^p\right)^{\frac{q}{2}}\right)^{\frac{2}{q}}\right). \tag{7}$$

that gives

$$\sup_{B_R(0)} u \leqslant C\left(R^{-\frac{2}{n}} \|u\|_{L^n(B_{2R}(0))} + \frac{1}{\varepsilon^2} R^{\frac{-4}{q}} \left(\|u\|_{L^{\frac{q}{2}}(B_{2R}(0))} + K\|u\|^p_{L^{\frac{pq}{2}}(B_{2R}(0))}\right)\right) \tag{8}$$

where C depends only on q and n. Now we apply Theorem 8.18 of [8] for $Lu = \varepsilon^2 \Delta u - f'(0)u$, the equation $Lu \leqslant 0$ and $R = \pi$. We get a constant $C > 0$, that depends on n and on $\frac{R}{\varepsilon}$ such that for all non-negative u satisfying $Lu \leqslant 0$ we have

$$R^{-\frac{2}{n}} \|u\|_{L^n(B_{2R}(0))} \leqslant C \inf_{B_R(0)} u. \tag{9}$$

But the constant C is a decreasing function of ε. Indeed, if $\varepsilon_1 < \varepsilon_2$ and if $\varepsilon_2^2\Delta u - f'(0)u \leqslant 0$, then $\varepsilon_1^2\Delta u - f'(0)u \leqslant 0$. So, if $C(\varepsilon_1)$ and $C(\varepsilon_2)$ are the best constants in (9), respectively for ε_1 and ε_2, we have $C(\varepsilon_2) \leqslant C(\varepsilon_1)$. On the other hand we have $\sup_{B_R(0)} u = \sup_{S^1\times\mathbb{R}_+} u$ and $\inf_{B_R(0)} u \leqslant \inf_{S^1\times\mathbb{R}_+} u$. Combining (8) and (9), we get (6).

3. Proof of Theorem 1.1

We may suppose that u is even in x_2 and consequently that $\frac{\partial u}{\partial x_2}(x_1,0) = 0$ for all $x_1 \in S^1$. Let us define $\Psi(x_2) = \frac{1}{2\pi}\int_0^{2\pi} u(x_1,x_2)\,\mathrm{d}x_1$. Integrating (2) on $[0,2\pi]$ we obtain

$$-\varepsilon^2\Psi''(x_2) + \frac{1}{2\pi}\int_0^{2\pi} f(u)\,\mathrm{d}x_1 = 0.$$

The hypotheses on f give

$$-\varepsilon^2\Psi''(x_2) \geqslant -f\big(\Psi(x_2)\big).$$

By the decaying property of u in x_2, we have that $\Psi'(x_2) < 0$. Multiplying by Ψ', integrating on $[0,+\infty[$ and using the Neumann condition on u we get

$$F\big(\Psi(0)\big) \geqslant 0, \tag{10}$$

where $F(u) = \int_0^u f(t)\,\mathrm{d}t$. It follows from the assumptions on f that F tends to $-\infty$ when u tends to $+\infty$. Let $C_\star$ be such that $F(u)$ is non-positive for $u > C_\star$. We have

$$\Psi(0) \leqslant C_\star, \tag{11}$$

that leads to $\inf_{x_1\in S^1} u(x_1,0) \leqslant C_\star$ and then, thanks to (6), for $\varepsilon \geqslant \varepsilon_1$, where $\varepsilon_1 > 0$ is given, we have

$$\sup_{S^1\times\mathbb{R}^+} u \leqslant C, \tag{12}$$

where C depends on ε_1 and is valid for any solution u of (2). Now we multiply (2) by $\frac{\partial u}{\partial x_2}$ and we integrate on $S^1\times\mathbb{R}_+$. We obtain

$$\frac{\varepsilon^2}{2}\int_0^{2\pi}\left(\frac{\partial u}{\partial x_1}(x_1,0)\right)^2\mathrm{d}x_1 + \int_0^{2\pi} F\big(u(x_1,0)\big)\,\mathrm{d}x_1 = 0. \tag{13}$$

Using (10) we get

$$\frac{\varepsilon^2}{2}\int_0^{2\pi}\left(\frac{\partial u}{\partial x_1}\right)^2(x_1,0)\,\mathrm{d}x_1 \leqslant \int_0^{2\pi}\Big(-F\big(u(x_1,0)\big) + F\big(\Psi(0)\big)\Big)\,\mathrm{d}x_1,$$

that leads to

$$\frac{\varepsilon^2}{2}\int_0^{2\pi}\left(\frac{\partial u}{\partial x_1}\right)^2(x_1,0)\,\mathrm{d}x_1 \leqslant -\int_0^{2\pi}\Big(F\big(u(x_1,0)\big) - F\big(\Psi(0)\big) - \big(u(x_1,0)-\Psi(0)\big)f\big(\Psi(0)\big)\Big)\,\mathrm{d}x_1. \tag{14}$$

However, by (11) and (12), given $\varepsilon_1 > 0$, there exists $M > 0$ such that $|f'(v)| \leqslant M$ for all v between $\Psi(0)$ and $u(x_1,0)$, $x_1 \in S^1$. Thus we have, for all $x_1 \in S^1$ and for all $\varepsilon > \varepsilon_1$,

$$\Big|F\big(u(x_1,0)\big) - F\big(\Psi(0)\big) - \big(u(x_1,0)-\Psi(0)\big)f\big(\Psi(0)\big)\Big| \leqslant M\big|u(x_1,0)-\Psi(0)\big|^2. \tag{15}$$

On the other hand the Poincaré inequality gives

$$\int_0^{2\pi}\big(u(x_1,0)-\Psi(0)\big)^2\,\mathrm{d}x_1 \leqslant 4\pi^2\int_0^{2\pi}\left(\frac{\partial u}{\partial x_1}\right)^2(x_1,0)\,\mathrm{d}x_1. \tag{16}$$

We deduce from (14)–(16) that there exists $C > 0$ such that for all $\varepsilon > \varepsilon_1$,

$$\frac{\varepsilon^2}{2}\int_0^{2\pi}\big(u(x_1,0)-\Psi(0)\big)^2\,\mathrm{d}x_1 \leqslant C\int_0^{2\pi}\big(u(x_1,0)-\Psi(0)\big)^2\,\mathrm{d}x_1.$$

This inequality gives that there exists $\bar\varepsilon > 0$ such that for $\varepsilon > \bar\varepsilon$ any solution of (2) verifies $\frac{\partial u}{\partial x_1}(x_1,0)=0$, for all $x_1 \in [0,2\pi]$. Let us prove that such a solution verifies in fact $\frac{\partial u}{\partial x_1}(x_1,x_2)=0$, for all $x_1 \in [0,2\pi]$ and for all $x_2 \in [0,+\infty[$. As f is twice differentiable in $]0,+\infty[$, except at isolated points, we may argue as follows. By derivation of (2) we get

$$-\varepsilon^2\Delta\frac{\partial u}{\partial x_1} + f'(u)\frac{\partial u}{\partial x_1} = 0. \tag{17}$$

Then we multiply this equation by $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ and we integrate on $S^1 \times \mathbb{R}+$. We obtain

$$\int_0^{+\infty}\int_0^{2\pi} f''(u)\frac{\partial u}{\partial x_2}\left(\frac{\partial u}{\partial x_1}\right)^2 \mathrm{d}x_1\,\mathrm{d}x_2 = 0.$$

However, f is concave and u decreases with respect to x_2, so we have $\frac{\partial u}{\partial x_1} = 0$ in $S^1 \times \mathbb{R}+$.

Remark 1. For $N > 2$, the positive solutions of (2) are radially symmetric and decreasing in $r = |(x_2,\dots,x_N)|$. But our above proof does not work for x_2 replaced by r. In this case we are unable to prove (10) because the equation for ψ is not the same one.

Acknowledgement

We thank Y. Ge for helpful discussions about this work.

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