Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 345 (2007) 289–292
COMPTES RENDUS MATHEMATIQUE
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Probability Theory/Functional Analysis

A new approach to Kolmogorov equations in infinite dimensions and applications to the stochastic 2D Navier–Stokes equation

Michael Röckner, Zeev Sobol

Department of Mathematics, Purdue University, West Lafayette, IN 47907-2, USA

Received 20 December 2006; accepted after revision 19 July 2007

Available online 23 August 2007

Presented by Paul Malliavin

Abstract

In this Note we present a new approach to solve Kolmogorov equations in infinitely many variables in weighted spaces of weakly continuous functions, including the case of non-constant possibly degenerate diffusion coefficients. ***To cite this article: M. Röckner, Z. Sobol, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***
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Résumé

Une nouvelle approche dans une infinité de dimensions et applications à l'équation Navier–Stokes stochastique en 2D. Dans cette Note nous présentons une nouvelle approche pour résoudre des équations de Kolmogorov à une infinité de variables dans des espaces à poids de fonctions faiblement continus. Le cas de coéfficients de diffusion non-constants et éventuellement dégénérés est inclus. ***Pour citer cet article : M. Röckner, Z. Sobol, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***
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The purpose of this Note is to present a new general approach to Kolmogorov equations in infinite dimensions based on the methods first developed in [2]. We illustrate this approach through its application to the stochastic 2D Navier–Stokes equations (NSE, see [1] and the references therein) with state dependent ('multiplicative') noise, which on an open set $\Omega \subseteq \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$ is given by

$$\frac{\partial}{\partial t}u + u \cdot \nabla u = \nu \Delta u - \nabla p + f, \quad \operatorname{div} u = 0, \quad u \restriction_{\partial\Omega} = 0, \quad u(x,0) = u_0(x). \tag{1}$$

Here $u(t,x) \in \mathbb{R}^2$ is the velocity of a fluid in $x \in \Omega$ at time $t \geqslant 0$, $p(t,x)$ the pressure, $f(t,x)$ an external stochastic force and ν the viscosity constant. We consider the Laplacian with Dirichlet and periodic boundary conditions.

As usual we project (1) onto the sub-space $H \subset L^2(\Omega \to \mathbb{R}^2)$ of divergence free vector fields by the Leray–Helmholtz projection P. Then the SPDE (1) becomes an SDE in H.

To describe the stochastic force f precisely, let $\{\ell_k\}_{k=1}^{\infty}$ be the eigenbasis of the part of Δ on H and let $\{w_t^k\}_{k=1}^{\infty}$ be a sequence of iid Brownian motions with $\mathcal{F}_t := \sigma\{w_t^k \mid 0 \leqslant s \leqslant t, k = 1, 2, 3, \ldots\}$ its associated filtration. If σ is an $(\mathcal{F}_t)$-adapted locally bounded separable process taking values in the space $L_2(H)$ of Hilbert–Schmidt operators on

E-mail addresses: roeckner@math.purdue.edu (M. Röckner), zsobol@gmail.com (Z. Sobol).

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doi:10.1016/j.crma.2007.07.009

H, the series $\sum_k \int_0^t \sigma \ell_k \, \mathrm{d}w_t^k$ converges in H almost surely. We denote the differential of the latter process by $\sigma \, \mathrm{d}w_t$ and set $f = \frac{\sigma(u)\, \mathrm{d}w_t}{\mathrm{d}t}$, with a continuous map $\sigma : H \to L_2(H)$, i.e. we allow σ to depend on the solution. Thus, (1) turns into the following SDE in H:

$$\mathrm{d}u_t = \big[\nu \Delta u_t - P(u_t \cdot \nabla u_t)\big]\, \mathrm{d}t + \sigma(u_t)\, \mathrm{d}w_t. \tag{2}$$

The usual way to obtain the Kolmogorov equations corresponding to SDE (2) is to reformulate the latter as a martingale problem, which is a standard approach to construct weak solutions to an SDE of type

$$\mathrm{d}u_t = \mu(u_t)\, \mathrm{d}t + \sigma(u_t)\, \mathrm{d}w_t \tag{3}$$

(cf. Stroock and Varadhan in [5] if $H = \mathbb{R}^d$): Let $\mathcal{D}$ be the set of all cylindrical functions of type

$$\Phi(u) = \phi\big(\langle \ell_1, u\rangle, \langle \ell_2, u\rangle, \ldots, \langle \ell_n, u\rangle\big), \quad n \in \mathbb{N}, \phi \in C_b^2\big(\mathbb{R}^n\big). \tag{4}$$

Itô's formula applied to $\Phi(u_t)$, with u_t solving (3), yields that

$$m_\Phi(t) := \Phi(u_t) - \Phi(u_0) - \int_0^t (L\Phi)(u_s)\, \mathrm{d}s, \tag{5}$$

is an $(\mathcal{F}_t)$-martingale, with the *Kolmogorov operator* L defined as follows:

$$L\Phi(u) = \frac{1}{2} \sum_{km} \big\langle \sigma(u)\ell_k, \sigma(u)\ell_m \big\rangle \frac{\partial^2 \Phi(u)}{\partial \ell_k \partial \ell_m} + \sum_k \mu_k(u) \frac{\partial \Phi(u)}{\partial \ell_k}, \quad \Phi \in \mathcal{D}, \tag{6}$$

where in the special case of (2)

$$\mu_k(u) := \big\langle \ell_k, \mu(u)\big\rangle = \langle \nu \Delta \ell_k, u\rangle + \langle u \cdot \nabla \ell_k, u\rangle, \quad k \in \mathbb{N}.$$

Then a solution to the *martingale problem* $(L, \mathcal{D})$ is a family of measures $(\mathbb{P}_u)_{u \in H}$ on $C([0,\infty), H)$, i.e. the space of continuous trajectories in H such that, for $u \in H$, first, $\mathbb{P}_u\{u_0 = u\} = 1$, and second, for $\Phi \in \mathcal{D}$, the process m_Φ is a $\mathbb{P}_u$-martingale with respect to the standard filtration on $C([0,\infty), H)$.

We confine ourselves to Markov solutions, i.e. $(\mathbb{P}_u)_{u \in H}$ form a Markov process. Then it suffices to construct the transition probability semigroup (TPS), i.e. a semi-group of Markov kernels $p_t(u, \mathrm{d}v)$ on H such that

$$p_t\Phi(u) - \Phi(u) = \int_0^t p_s(L\Phi)(u)\, \mathrm{d}s, \quad t > 0, \Phi \in \mathcal{D}, \tag{7}$$

which is obtained from (5) by taking expectation. (7) as equations in the unknown measures $p_s(u, \mathrm{d}v)$ are called *Kolmogorov equations* and by construction can be considered as a linearization of (3).

A purely analytic method of solving (7) was introduced in [2] and then developed in [3] (see also [4]). Its main point is the construction of the TPS p_t as a semi-group P_t of Markov operators on

$$C_{\mathbb{V}} := \Big\{ f : \{\mathbb{V} < \infty\} \to \mathbb{R} \mid \\ f\restriction_{\{\mathbb{V} \leqslant R\}} \text{ is weakly continuous } \forall R > 0 \text{ and } \lim_{R\to\infty} \sup_{\{\mathbb{V} \geqslant R\}} \mathbb{V}^{-1}|f| = 0 \Big\}, \tag{8}$$

$\mathbb{V} : H \to [0,\infty]$ being a Lyapunov function for L, i.e. $\mathbb{V}$ is of compact level sets, such that $(\lambda - L)\mathbb{V} > 0$.

To state our result precisely, let us consider the SDE (3) on an abstract separable Hilbert space H. Let $H_n \subset H_{n+1} \subset H$, be an increasing sequence of finite dimensional subspaces of H, $H_\infty := \bigcup H_n$ be dense in H, $P_n : H \to H_n$ be the corresponding orthogonal projections.

Hypothesis 1. The noise $\sigma : H \to L_2(H)$ is Lipschitz continuous and has block diagonal structure, that is, there exists a sequence $N_n \to \infty$ such that $P_{N_n}\sigma(u) = P_{N_n}\sigma(P_{N_n}u)$ for all $u \in H$.

Hypothesis 2. Let $N_n \to \infty$ be as in Hypothesis 1, $\sigma_n(u) := P_{N_n}\sigma(u) = P_{N_n}\sigma(P_{N_n}u)$. For all $n \in \mathbb{N}$, there exist $\mu_n \in C(H \to H_{N_n})$, and $\mathbb{V}_n \in C^2(H)$, $\mu_n(u) = \mu_n(P_{N_n}u)$, $\mathbb{V}_n(u) = \mathbb{V}_n(P_{N_n}u)$ for all $u \in H$, such that

(a) $\mathbb{V}_n > 0$;
(b) $\sup_{u,w\in H_{N_n},\,u\neq w,\,|u|,|w|\leqslant R} \frac{\langle \mu_n(u)-\mu_n(w),u-w\rangle}{|u-w|^2} < \infty$;
(c) There exists $\lambda\in\mathbb{R}$ independent of n such that, for a.a. $u\in H_{N_n}$,

$$\limsup_{H_{N_n}\ni w\to u} \frac{\langle \mu_n(u)-\mu_n(w),u-w\rangle}{|u-w|^2} + \sup_{\xi\in H_{N_n},|\xi|=1} |D_\xi\sigma_n|^2_{L_2}(u) + \sup_{\xi\in H_{N_n},|\xi|=1}\left\langle D_\xi\sigma_n^*(x)\xi,\sigma_n^*\frac{D\mathbb{V}_n}{\mathbb{V}_n}\right\rangle(u) + \frac{L_n\mathbb{V}_n}{\mathbb{V}_n}(u)\leqslant\lambda, \tag{9}$$

where L_n on $C^2(H_{N_n})$ is given by (6) with μ_n,σ_n replacing σ and μ, respectively.

Hypothesis 3. Let $N_n\to\infty$ be as in Hypothesis 1, and $\mu_n,\mathbb{V}_n,L_n$ be as in Hypothesis 2. There are positive functions $\mathbb{V},\mathbb{W}$ of compact level sets, finite on H_∞, such that

(a) $\mathbb{V}_n,\mathbb{V}\in C_{\mathbb{W}}$ (the latter is defined as in (8)) and $\mathbb{V}_n\to\mathbb{V}$ in $C_{\mathbb{W}}$ as $n\to\infty$;
(b) For all $u\in\{\mathbb{W}<\infty\}$, $\mu(u)$ is defined, $|\mu_n-P_{N_n}\mu|(u)\leqslant c\frac{\mathbb{W}}{\mathbb{V}}(u)$ and $|\mu_n-P_{N_n}\mu|(u)\to 0$ as $n\to\infty$;
(c) $\limsup_{n\to\infty}\inf_{u\in H_{N_n}}\frac{(\lambda_*-L_n)\mathbb{V}_n}{\mathbb{W}}(u)\geqslant 1$ for some $\lambda_*\in\mathbb{R}$.

The following theorem is our main result in [3]. To the best of our knowledge it is the first result on solving the Kolmogorov equations (7) purely analytically for all points u in an explicitly specified subspace of H and with a non-constant possibly degenerate diffusion matrix in the second order part of L.

Theorem 4. *Let Hypotheses* 1–3 *hold. Then there exists a unique solution to* (7) *on* $\{\mathbb{V}<\infty\}$ *and the TPS constitutes a* C_0*-semi-group of quasi-contractions on* $C_{\mathbb{V}}$*. Furthermore, there exists a unique Markov solution* $(\mathbb{P}_u)_{u\in\{\mathbb{V}<\infty\}}$ *of* (5).

We now apply Theorem 4 to the 2D NSE (2). Let H be the sub-space of $L^2(\Omega\to\mathbb{R}^2)$ consisting of all divergence free vector fields, let $H_0^1:=H_0^1(\Omega\to\mathbb{R}^2)$ (note that $H_0^1=H^1$ if $\Omega=\mathbb{T}^2$), $H^2:=H^2(\Omega\to\mathbb{R}^2)$ and let $\mu(u):=\nu\Delta u-P(u\cdot\nabla u)$ for $u\in H_0^1\cap H^2$.

Theorem 5. *Let* $\sigma: H\to L_2(H,H_0^1)$ *be bounded, satisfying Hypothesis* 1.

Moreover, let $\mathbb{V}(u)=\mathbb{V}_\varkappa(u)=\mathrm{e}^{\varkappa|\nabla u|^2}$ *for* $\varkappa<\nu/\sup_u|\sigma(u)|^2_{H\to H}$.

Then (7) *for L with* μ *and* σ *as above has a unique solution on* $H_0^1\cap H$ *and the respective TPS constitutes a* C_0*-semi-group of quasi-contractions on* $C_{\mathbb{V}}$*. Furthermore, there exists a unique Markov solution* $(\mathbb{P}_u)_{u\in H_0^1\cap H}$ *of the corresponding martingale problem.*

Proof. Let $\mathbb{W}(u):=c\mathbb{V}(u)|\Delta u|^2$ if $u\in H_0^1\cap H^2$, and $\mathbb{W}\equiv+\infty$ else. Let H_n be the linear hull of the first n eigenvectors of Δ, $\mathbb{V}_n(u):=\mathbb{V}(P_nu)$ and $\mu_n(u):=P_n\mu(P_nu)$, $n\in\mathbb{N}$. Then $|P_n\mu(u)-\mu_n(u)|\leqslant 2|u||\nabla u|\leqslant c|\Delta u|^2$. So Hypothesis 2(a)–(b) and Hypothesis 3(a)–(b) readily follow.

Note that for $u,\xi,\eta\in H\cap H_0^1\cap H^2$

$$\frac{D_\xi\mathbb{V}}{\mathbb{V}}(u)=-2\varkappa\langle\Delta u,\xi\rangle,\qquad \frac{D^2_{\xi\eta}\mathbb{V}}{\mathbb{V}}(u)=4\varkappa^2\langle\Delta u,\xi\rangle\langle\Delta u,\eta\rangle-2\varkappa\langle\Delta\xi,\eta\rangle,$$
$$\big\langle\Delta u,P(u\cdot\nabla u)\big\rangle=\int_\Omega(\operatorname{curl}u)\operatorname{curl}P(u\cdot\nabla u)\,\mathrm{d}s=\int_\Omega(\operatorname{curl}u)(u\cdot\nabla\operatorname{curl}u)\,\mathrm{d}s=0.$$

So

$$\begin{aligned}\frac{L_n\mathbb{V}_n}{\mathbb{V}_n}(u)&=-2\varkappa\nu|\Delta u|^2+2\varkappa^2\big|\sigma^*(u)\Delta u\big|^2+\varkappa\big|\sigma^*(u)(-\Delta)^{1/2}\big|^2_{L^2(H)}\\&\leqslant-2\varkappa\big(\nu-\varkappa\sup_u\big|\sigma(u)\big|^2_{H\to H}\big)|\Delta u|^2+C.\end{aligned}\tag{10}$$

So Hypothesis 3(c) follows. Furthermore, for $u, w \in H_0^1 \cap H^2 \cap H$,

$$\bigl\langle u - w, P(u \cdot \nabla u) - P(w \cdot \nabla w)\bigr\rangle = \int_\Omega (u - w) \cdot (u \cdot \nabla u - w \cdot \nabla w)\,\mathrm{d}s$$

$$= \int_\Omega (u - w) \cdot \bigl((u - w) \cdot \nabla u\bigr)\,\mathrm{d}s,$$

since $\int_\Omega (u - w) \cdot (w \cdot \nabla(u - w))\,\mathrm{d}s = \frac{1}{2}\int_\Omega w \cdot \nabla|u - w|^2\,\mathrm{d}s = 0$.

So, $|\langle u - w, P(u \cdot \nabla u) - P(w \cdot \nabla w)\rangle| \leqslant |\Delta u||(-\Delta)^{-1/2}|u - w|^2| \leqslant c|\Delta u||u - w|^2$.

Hence, for any $\varkappa, \varepsilon > 0$,

$$\limsup_{H_{N_n} \ni w \to u} \frac{\langle \mu_n(u) - \mu_n(w), u - w\rangle}{|u - w|^2} \leqslant 2\varkappa\varepsilon|\Delta u|^2 + \frac{c}{\varkappa\varepsilon}.$$

Now, using (10) it is easy to verify (9) and thus Hypothesis 2(c) holds. □

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