Partial Differential Equations

Ill-posedness of H^1-supercritical waves

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Abstract

We extend the results of Lebeau (2005) and Christ et al. (2007) to more general H^1-supercritical nonlinearities. We also extend those results to the 2D case for exponentially growing nonlinearities. The proof uses the finite speed of propagation and a quantitative study of the associated O.D.E. It does not require any scaling invariance of the equation. ***To cite this article: S. Ibrahim et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

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Résumé

Instabilité des ondes H^1-surcritiques. Nous étendons les résultats de Lebeau (2005) et Christ et al. (2007) aux cas de nonlinéarités surcritiques quelconques. On traite aussi le cas 2D pour des nonlinéarités à croissance exponentielle. La preuve utilise la vitesse finie de propagation et une étude quantitative de l'E.D.O. associée. ***Pour citer cet article : S. Ibrahim et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

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L'objectif de cette Note est d'étudier le caractère bien-posé d'une équation des ondes générique du type

$$\partial_t^2 u - \Delta u + F'(u) = 0, \quad \text{dans} \quad \mathbb{R}_t \times \mathbb{R}_x^N, \tag{1}$$

où $N \geqslant 2$ et $F : \mathbb{R} \to \mathbb{R}$ est une fonction régulière *paire* vérifiant

$$F(0) = F'(0) = 0 \quad \text{et} \quad u F'(u) \geqslant 0. \tag{H1}$$

Rappellons que, sous l'hypothèse (**H1**), l'équation (1) admet au moins une solution faible globale à énergie finie dès que les données sont à énergie finie (voir, par exemple, [16,17]). Il est bien connu que si la dimension est plus grande que trois et si la nonlinéarité est au plus critique, alors le problème de Cauchy associé à (1) est globalement bien posé dans l'espace d'énergie (voir à ce propos [6,5,15]).

En dimension deux d'espace, la nonlinéarité H^1-critique semble être du type exponentiel et les premiers travaux dans cette direction sont ceux de Nakamura–Ozawa [13,14] dans lesquels un résultat d'existence globale à données petites a été établi. Dans un travail récent [7], les auteurs ont considéré le cas

$$F(u) = \frac{e^{4\pi u^2} - 1}{8\pi} \tag{2}$$

et ont établi l'existence globale dans l'espace d'énergie dans le cas au plus critique i.e. $E_0 \leqslant 1$ où E_0 est l'énergie totale initiale (voir [7] pour plus des détails). Dans le cas d'une non-linéarité polynomiale H^1-sur-critique, seuls des résultats d'instabilité sont connus. Voir par exemple [10,4] et [11]. On renvoie le lecteur à la version anglaise pour plus des détails. Tous ces travaux ne donnent pas une réponse à la question intéressante suivante (toujours concernant le cas sur-critique) : peut-on propager une régularité H^1 (ou même meilleur) globalement en temps ? Loin d'apporter de réponse à cette question, les résultats obtenus ici fournissent quelques indications concernant l'instabilité des solutions.

Dans toute la suite, on va supposer que l'équation (1) est H^1-sur-critique. Pour celà on fait sur la nonlinéarité F l'hypothèse suivante :

$$\begin{cases} \text{lorsque } N \geqslant 3, \quad \dfrac{F(u)}{|u|^{\frac{2N}{N-2}}} \quad \text{croit vers } +\infty, \quad u \to \infty, \\[4mm] \text{lorsque } N = 2, \quad \dfrac{\log(F(u))}{|u|^2} \quad \text{croit vers } +\infty, \quad u \to \infty. \end{cases} \tag{H2}$$

Nos résultats principaux sont contenus dans les théorèmes suivants :

Théorème 0.1. *Supposons que la nonlinéarité F satisfait les hypothèses* (**H1**), (**H2**). *Il existe une suite* (φ_k) *convergeant vers* 0 *dans* $\dot{H}^1(\mathbb{R}^N)$ *avec* $\sup_k E(\varphi_k) < \infty$, *et une suite de réels positifs* (t_k) *convergeant vers* 0 *telle que toute solution faible* u_k *de* (1) *de donnée* $(\varphi_k, 0)$ *satisfait* :

$$\liminf_{k \to \infty} \| \partial_t u_k(t_k) \|_{L^2} \gtrsim 1. \tag{3}$$

Ici $E(\varphi) = E(\varphi, 0)$ où $E(\varphi, \psi)$ désigne l'énergie totale de (φ, ψ) définie par

$$E(\varphi, \psi) \overset{\text{déf}}{=} \|\nabla\varphi\|_{L_x^2}^2 + \|\psi\|_{L_x^2}^2 + \int 2F(\varphi)\, dx. \tag{4}$$

Remarque 1. En 2D, si $\lim_{u \to \infty} \frac{\log(F(u))}{u^2} = 4\pi$ (ou plus généralement, égale à $\alpha > 0$), alors la criticalité est mesurée par la taille de l'énergie. Dans ce cas, le problème de Cauchy est dit sur-critique si $E_0 > 1$ (ou plus généralement $E_0 > \frac{4\pi}{\alpha}$).

Le second résultat complète les résultats dans [7] concernant le cas sur-critique en montrant la non continuité uniforme du flot si $E_0 > 1$:

Théorème 0.2. *On suppose que la nonlinéarité F est donnée par* (2). *Il existe une suite de réels positifs* (t_k) *tendant vers zéro et une suite de fonctions* (f_k) *dans* $H^1(\mathbb{R}^2)$ *vérifiant, pour tout* $v > 0$,

$$\left\| f_k\left(\frac{\cdot}{v}\right) \right\|_{H^1} \lesssim v + 1, \quad 0 < E\left(\left(1 + \frac{1}{k}\right) f_k\left(\frac{\cdot}{v}\right)\right) - 1 \leqslant e^3 v^2 \quad et \quad 0 < E\left(f_k\left(\frac{\cdot}{v}\right)\right) - 1 \leqslant v^2.$$

De plus, pour toutes solutions faibles u_k^v *et* v_k^v *de* (1) *de données* $((1 + \frac{1}{k}) f_k(\frac{\cdot}{v}), 0)$ *et* $(f_k(\frac{\cdot}{v}), 0)$ *respectivement, on a*

$$\liminf_{k \longrightarrow \infty} \left\| \partial_t \left(u_k^v - v_k^v\right)(t_k) \right\|_{L^2} \gtrsim v.$$

Nous renvoyons à [8] pour des détails et des compléments.

1. Introduction

In this Note, we study the well-posedness of the Cauchy problem associated to the semilinear wave equation

$$\partial_t^2 u - \Delta u + F'(u) = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x^N, \tag{5}$$

where $N \geqslant 2$ and $F : \mathbb{R} \to \mathbb{R}$ is an *even* regular function satisfying hypothesis (**H1**) and (**H2**) described above.

With hypothesis (**H1**), one can construct a global weak solution with finite energy data using a standard compactness argument (see, for example, [16]). The second assumption (**H2**) means that our Cauchy problem is H^1-supercritical.

For any weak solution of (5) we define the total energy by

$$E(u(t)) \stackrel{\text{def}}{=} \|\nabla_{t,x} u(t)\|_{L_x^2}^2 + \int_{\mathbb{R}^N} 2F(u(t)) \, dx.$$

The energy of the data $(\varphi, \psi) \in \dot{H}^1 \times L^2$ is given by (4). When $\psi = 0$ we denote $E(\varphi, 0) = E(\varphi)$.

Let us recall a few historic facts about this problem. When the space dimension $N \geqslant 3$, the defocussing semilinear wave equation with power p reads

$$\partial_t^2 u - \Delta u + |u|^{p-1} u = 0, \tag{6}$$

where $p > 1$. This problem has been widely investigated and there is a large literature dealing with the well-posedness theory of (6) in the scale of the Sobolev spaces H^s. Concerning the global solvability in the energy space $\dot{H}^1 \times L^2$, there are mainly three cases. The first case is when $p < p_c$ where $p_c = \frac{N+2}{N-2}$, this is the subcritical case. In this case, Ginibre and Velo [5], showed that the problem (6) is globally well-posed in the energy space. Second, if the exponent p is critical, (which means $p = p_c$), this problem was solved by Shatah–Struwe ([15] and references therein). Finally, for H^1-supercritical polynomial nonlinearities, Lebeau [10] has proved an instability phenomena for real valued radially symmetric solutions. The same question has been dealt in Christ–Colliander–Tao's paper [4]. It was shown that the solution map fails to be continuous at the zero initial data with respect to the H^1 topology, and it is not uniformly continuous with respect to the stronger topology defined by the total energy. A much stronger result was given in [11] where Lebeau constructed a solution to the heptic nonlinear wave equation which immediately looses its H^s regularity for all $s > 1$. More recently, Burq–Gérard–Ibrahim [2] extended the results of [4] to more general supercritical nonlinearities and other nonflat geometries for initial data in H^s with $s > 1$. Similar results were obtained for the nonlinear Schrödinger equation. Among others, we mainly cite the works [3,9,4] and [1].

In dimension two, H^1-critical nonlinearities seems to be of exponential type,[1] since every power is subcritical. In a recent work [7], the case $F(u) = (e^{4\pi u^2} - 1)/(8\pi)$ was investigated and the criticality was proposed with respect to the size of the energy. In fact, global well-posedness result was obtained in both subcritical ($E_0 < 1$) and critical cases ($E_0 = 1$). Here we intend to complete these results by showing a kind of instability in the supercritical case. Our proof also extends to all dimensions $N \geqslant 2$, and for non-polynomial nonlinearities. Precisely, we have the following:

Theorem 1. *We assume that* (**H1**) *and* (**H2**) *are fulfilled. There exists a sequence* (φ_k) *in* $\dot{H}^1$ *and a sequence* (t_k) *in* $(0, \infty)$ *satisfying*

$$\|\nabla \varphi_k\|_{L_x^2} \longrightarrow 0, \quad t_k \longrightarrow 0, \quad \sup_k E(\varphi_k) < \infty,$$

and such that any weak solution u_k *of* (5) *with initial data* $(\varphi_k, 0)$ *satisfies*

$$\liminf_{k \to +\infty} \|\partial_t u_k(t_k)\|_{L_x^2} \gtrsim 1.$$

The second result concerns the case where F is given by (2). We will show the ill-posedness in the supercritical case which reveals the optimality of the authors earlier work [7].

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[1] In fact, the critical nonlinearity is of exponential type in any dimension N with respect to $H^{N/2}$ norm.

Theorem 2. *We assume that the nonlinearity F is given by (2). There exists a sequence of positive real numbers (t_k) going to zero and a sequence of functions (f_k) in $H^1(\mathbb{R}^2)$ such that, for any $v > 0$,*

$$\left\| f_k\left(\frac{\cdot}{v}\right) \right\|_{H^1} \lesssim v + 1, \quad 0 < E\left(\left(1 + \frac{1}{k}\right) f_k\left(\frac{\cdot}{v}\right)\right) - 1 \leqslant e^3 v^2 \quad and \quad 0 < E\left(f_k\left(\frac{\cdot}{v}\right)\right) - 1 \leqslant v^2. \tag{7}$$

Moreover, for every weak solutions u_k^v and v_k^v of (1) with initial data $((1 + \frac{1}{k})f_k(\frac{\cdot}{v}), 0)$ and $(f_k(\frac{\cdot}{v}), 0)$ respectively, we have

$$\liminf_{k \to \infty} \left\| \partial_t \left(u_k^v - v_k^v\right)(t_k) \right\|_{L^2} \gtrsim v.$$

2. Sketches of the proofs

In this section we will give only the ideas of the proofs. We refer to [8] for more details and some complements. We begin by Theorem 1.

The case $N \geqslant 3$: For $k \geqslant 1$ define the function φ_k by

$$\varphi_k(x) = 0 \quad \text{if } |x| \geqslant 1, \quad \varphi_k(x) = a(k, \varepsilon)\left(|x|^{2-N} - 1\right) \quad \text{if } \frac{\varepsilon}{k} \leqslant |x| \leqslant 1, \quad \varphi_k(x) = k^{\frac{N-2}{2}} \quad \text{if } |x| \leqslant \frac{\varepsilon}{k},$$

where $\varepsilon = \varepsilon_k \stackrel{\text{def}}{=} k(F(k^{\frac{N-2}{2}}))^{-\frac{1}{N}}$ and $a(k, \varepsilon) \sim (\varepsilon/\sqrt{k})^{N-2}$ is chosen such that φ_k is continuous. An easy computation yields

$$\|\nabla \varphi_k\|_{L^2}^2 \lesssim \varepsilon^{N-2}.$$

Using assumption (**H2**), we get

$$\int_{\mathbb{R}^N} F\left(\varphi_k(x)\right) dx \lesssim F\left(k^{\frac{N-2}{2}}\right)\left(\frac{\varepsilon}{k}\right)^N + \int_{\frac{\varepsilon}{k}}^1 F\left(a(k, \varepsilon)\left(r^{2-N} - 1\right)\right) r^{N-1} dr$$

$$\lesssim F\left(k^{\frac{N-2}{2}}\right)\left(\frac{\varepsilon}{k}\right)^N \left(1 + \frac{1 - (\frac{\varepsilon}{k})^N}{(1 - (\frac{\varepsilon}{k})^{N-2})^{\frac{2N}{N-2}}}\right) \lesssim C,$$

and hence $\sup_k E(\varphi_k) < \infty$. Now, consider the ordinary differential equation associated to (5)

$$\ddot{\Phi} + F'(\Phi) = 0, \qquad \left(\Phi(0), \dot{\Phi}(0)\right) = \left(k^{\frac{N-2}{2}}, 0\right). \tag{8}$$

Using the assumptions on F, we can see that (8) has a unique global smooth and periodic solution Φ_k with period

$$T_k = 2\sqrt{2} \int_0^{k^{\frac{N-2}{2}}} \frac{d\Phi}{\sqrt{F(k^{\frac{N-2}{2}}) - F(\Phi)}} = 2\sqrt{2} \frac{k^{\frac{N-2}{2}}}{\sqrt{F(k^{\frac{N-2}{2}})}} \int_0^1 \left(1 - \frac{F(vk^{\frac{N-2}{2}})}{F(k^{\frac{N-2}{2}})}\right)^{-1/2} dv.$$

Using assumption (**H2**), we have

$$T_k \leqslant 2\sqrt{2} \frac{k^{\frac{N-2}{2}}}{\sqrt{F(k^{\frac{N-2}{2}})}} \int_0^1 \left(1 - v^{\frac{2N}{N-2}}\right)^{-1/2} dv \lesssim k^{\frac{N-2}{2}} \left(F\left(k^{\frac{N-2}{2}}\right)\right)^{-1/2} \ll \frac{\varepsilon_k}{k}.$$

Now we are in position to construct the sequence (t_k). First, recall that by finite speed of propagation, any weak solution u_k of (5) with data $(\varphi_k, 0)$ satisfy $u_k(t, x) = \Phi_k(t)$ if $0 < t < \frac{\varepsilon_k}{k}$ and $|x| < \frac{\varepsilon_k}{k} - t$. Then we choose $t_k = \frac{T_k}{4}$ such that $\Phi_k(t_k) = 0$. It follows that $t_k \ll \frac{\varepsilon_k}{k}$ and, for $|x| < \frac{\varepsilon_k}{k} - t_k$,

$$\left|\partial_t u_k(t_k, x)\right| = \sqrt{2}\sqrt{F\left(k^{\frac{N-2}{2}}\right) - F(\Phi_k(t_k))} \gtrsim \sqrt{F\left(k^{\frac{N-2}{2}}\right)}.$$

Hence

$$\left\| \partial_t u_k(t_k) \right\|_{L^2}^2 \gtrsim F\left(k^{\frac{N-2}{2}}\right)\left(\frac{\varepsilon_k}{k} - t_k\right)^N = C\left(\frac{\varepsilon_k}{k}\right)^N F\left(k^{\frac{N-2}{2}}\right)\left(1 - t_k\frac{k}{\varepsilon_k}\right)^N \gtrsim C,$$

and the conclusion follows.

The dimension two: For $k \geqslant 1$, we define φ_k by

$$\varphi_k(x) = 0 \quad \text{if } |x| \geqslant 1, \quad \varphi_k(x) = \frac{-2\sqrt{k}\log|x|}{\log(F(\sqrt{k}))} \quad \text{if } \varepsilon_k e^{-k/2} \leqslant |x| \leqslant 1, \quad \varphi_k(x) = \sqrt{k} \quad \text{if } |x| \leqslant \varepsilon_k e^{-k/2},$$

where $\varepsilon_k = e^{k/2}(F(\sqrt{k}))^{-1/2}$. Once the functions (φ_k) are given, the rest of the proof is the same as in higher dimension. We skip the details here (see [8]). □

Next, we return to Theorem 2. For any $k \geqslant 1$ define f_k by:

$$f_k(x) = 0 \quad \text{if } |x| \geqslant 1, \quad f_k(x) = -\frac{\log|x|}{\sqrt{k\pi}} \quad \text{if } e^{-k/2} \leqslant |x| \leqslant 1, \quad f_k(x) = \sqrt{\frac{k}{4\pi}} \quad \text{if } |x| \leqslant e^{-k/2}.$$

These functions were introduced in [12] to show the optimality of the exponent 4π in Trudinger–Moser inequality. An easy computation shows that $\|\nabla f_k\|_{L^2(\mathbb{R}^2)} = 1$ and $\|f_k\|_{L^2(\mathbb{R}^2)} \lesssim \frac{1}{\sqrt{k}}$. The assertion (7) follows directly from the following lemma:

Lemma 2.1. *For any $a \geqslant 1$ and $k \in \mathbb{N}$, we have*

$$\int_{e^{-\frac{k}{2}}}^{1} r e^{\frac{4a^2}{k}\log^2 r}\, \mathrm{d}r \leqslant 2e^{(a^2-1)k}.$$

Now, let Φ_k and Ψ_k be the solutions of the associated O.D.E with initial data $((1 + \frac{1}{k})\sqrt{\frac{k}{4\pi}}, 0)$ and $(\sqrt{\frac{k}{4\pi}}, 0)$ respectively. We can see that the period T_k of Φ_k satisfies $T_k \lesssim \sqrt{k}\, e^{-(1+\frac{1}{k})^2 k/2}$. Choose $t_k \in]0, T_k/4[$ such that

$$\Phi_k(t_k) = (1 + 1/k)\sqrt{\frac{k}{4\pi}} - \left((1 + 1/k)\sqrt{\frac{k}{4\pi}}\right)^{-1},$$

then $t_k \lesssim \frac{1}{\sqrt{k}}e^{-k/2} \lesssim \frac{v}{2}e^{-k/2}$ (for k large). To finish the proof, we use the finite speed of propagation and the fact that

$$\left|\left(\dot{\Phi}_k(t_k)\right)^2 - \left(\dot{\Psi}_k(t_k)\right)^2\right| = \frac{1}{4\pi}\left|\left(e^{4\pi\Phi_k(0)^2} - e^{4\pi\Phi_k(t_k)^2}\right) - \left(e^{4\pi\Psi_k(0)^2} - e^{4\pi\Psi_k(t_k)^2}\right)\right| \gtrsim e^k. \qquad \Box$$

Remark 1. 1) In Theorem 1, if we relax the condition $\sup_k E(\varphi_k) < \infty$, we can construct a sequence φ_k such that $\|\nabla\varphi_k\|_{L^2_x} \to 0$, $E(\varphi_k) \to \infty$ and $\lim\|\partial_t u_k(t_k)\|_{L^2_x} = \infty$, $\lim\|\nabla_x u_k(t_k)\|_{L^2_x} = \infty$.

2) We can regularize φ_k and hence obtain results in H^s for $s \geqslant 1$ if the equation is supercritical in H^s (see also the first example below).

3. Some examples

1. Let m be an integer with $m \geqslant 3$. Consider the following three-dimensional wave equation:

$$\partial_t^2 u - \Delta u + u^{2m+1} = 0.$$

In [10], Lebeau proves the nonuniform continuity with respect to the initial data in H^1. Also, in [11] Lebeau proves the loss of regularity in H^s for every $s \in]1, \frac{3}{2} - \frac{1}{m}[$ (see also [4]). Here, we recover similar results.

2. Let α be a nonnegative real number and consider the three-dimensional wave equation

$$\partial_t^2 u - \Delta u + u^5\left(\log(2 + u^2)\right)^\alpha = 0.$$

This equation is just barely energy-supercritical in the sense that the nonlinear potential satisfy

$$\int\limits_0^u v^5\big(\log\big(2+v^2\big)\big)^\alpha\, \mathrm{d}v \approx u^6\big(\log\big(2+u^2\big)\big)^\alpha.$$

When $\alpha = 1$ and in the radial case, Tao [18] proves the global existence of regular solutions.

Our theorem holds here and gives a sort of ill-posedness in H^1.

3. Let $m \geqslant 2$ be an integer and consider the following 2D semilinear wave equation

$$\partial_t^2 u - \Delta u + u^{2m-1}\mathrm{e}^{4\pi u^{2m}} = 0.$$

The nonlinearity satisfies the assumptions (**H1**), (**H2**). $\square$

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