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Differential Topology

Fundamental group of discriminant complements of Brieskorn–Pham polynomials

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Abstract

In this Note we recall the braid monodromy of discriminants of hypersurface singularities and present two results from Lönne (2003): the braid monodromy associated to hypersurface singularities of Brieskorn–Pham type is given explicitly in terms of finitely many braids, and we show how this leads to very nice finite presentations of fundamental groups of the discriminant complements. ***To cite this article: M. Lönne, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Groupe fondamental du complément de la discriminante des singularités de Brieskorn–Pham. En cette Note nous rapellons la monodromie de tresses pour le discriminants de singularités d'une hypersurface et nous présentons deux résultats de Lönne (2003) : la monodromie de tresses associée aux singularités de Brieskorn–Pham est donnée par un nombre fini de tresses, et nous en déduisons une très belle présentation finie du groupe fondamental d'un complément d'une discriminante. ***Pour citer cet article : M. Lönne, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***
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1. Introduction

The topic of discriminant complement of hypersurface singularities has attracted much attention since long ago. In the case of simple singularities such complements are identified as spaces of regular orbits for the Weyl group of the same type and are shown to be aspherical. Their fundamental groups are given by the Artin–Brieskorn groups of the same type with a natural finite presentation encoded by the corresponding Dynkin diagram. So there is a strong link to natural combinatorial structures.

Sadly enough there has not been much progress towards a solution of the problems, formulated by Brieskorn in [3], which he intended as guidelines to the case of more general singularities. There he asked explicitly for the fundamental group and suggested obtaining such groups from a generic plane section using the theorem of Zariski and van Kampen, cf. [16,10].

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In fact our recent achievement [11] in case of Brieskorn–Pham polynomials is based on this idea, but relies heavily on the investigation of non-generic plane section and their relation to generic ones.

2. Braid monodromy

First recall that a holomorphic function f, more precisely a holomorphic function germ, is studied by means of versal unfoldings, e.g. given by a function

$$F(x,z,u) = f(x) - z + \sum b_i u_i,$$

such that the b_i and the Jacobian ideal of f span the ideal of holomorphic function germs vanishing at the origin.

In case of a semi-universal unfolding the unfolding dimension is given by the Milnor number $\mu = \mu(f)$ and we get a diagram

$$\begin{array}{ccccc} (z,u_1,\ldots,u_{\mu-1}) \in & \mathbf{C}^\mu & \supset \mathcal{D} & = & \{(z,u) \mid F_{z,u}^{-1}(0) \text{ is singular}\} \\ \downarrow & {}^{p}\!\downarrow & \downarrow & & \\ (u_1,\ldots,u_{\mu-1}) \in & \mathbf{C}^{\mu-1} & \supset \mathcal{B} & = & \{u \mid F_{0,u} \text{ is not Morse}\}. \end{array}$$

The restriction $p|_{\mathcal{D}}$ of the projection to the discriminant $\mathcal{D}$ is a finite map, such that the branch set coincides with the bifurcation set $\mathcal{B}$.

The key step is to find a natural fibre bundle structure for the restriction of p to $p^{-1}(\mathbf{C}^{\mu-1} \setminus \mathcal{B}) \setminus \mathcal{D}$. To cope with the inherent local character of our set-up we further restrict to a local neighbourhood which is a closed disc bundle over a contractible base, such that all disc boundaries are disjoint from $\mathcal{D}$. We thus get a fibre bundle of μ-punctured discs with trivialisable boundary. Hence the monodromy takes values in the group of classes of diffeomorphisms up to isotopy fixing the boundary.

This approach follows the idea of Enriques and Zariski, which they used in their study of covers branched along plane curves [4,16], and which has been successfully revived by Moishezon [12] under the notion of *braid monodromy*.

The theory of polynomial coverings of Hansen, [8], provides another framework to get a structure homomorphism from the fundamental group of the base to the group of classes of diffeomorphisms of the μ-punctured disc fibre up to isotopy fixing the boundary, which is naturally identified with the braid group $\mathbf{Br}_\mu$ up to inner automorphisms. The same map is induced by the Lyashko–Looijenga classifying map to the space $\mathbf{C}^\mu$ of monic polynomials of degree μ, which takes the complement of the bifurcation set to the complement of the set Δ of polynomials with multiple roots.

Definition 2.1. The *braid monodromy homomorphism* is the map

$$LL_*: \pi_1(\mathbf{C}^{\mu-1} \setminus \mathcal{B}) \to \pi_1(\mathbf{C}^\mu \setminus \Delta) \cong \mathbf{Br}_\mu,$$

(the fundamental group of the left is to be understood in the local, i.e. in the germ sense).

To determine the fundamental group of the discriminant complement, we only need a weaker invariant:

Definition 2.2. The *braid monodromy group* is the image $\operatorname{im} LL_* \subset \mathbf{Br}_\mu$, which up to conjugation is a well-defined invariant of the right equivalence class of f.

3. Brieskorn–Pham polynomials

By adding inductively a pure monomial in an additional variable to the pure monomial corresponding to a type A singularity we get polynomials, which figure prominently in the work of Brieskorn and Pham:

Definition 3.1. A polynomial $f \in \mathbf{C}[x_1,\ldots,x_n]$ is called *Brieskorn–Pham polynomial*, if with $l_i \in \mathbf{Z}^{>0}$

$$f(x_1,\ldots,x_n) = x_1^{l_1+1} + \cdots + x_n^{l_n+1}.$$

To our ends we recall a result of which Pham, Gabrielov and Hefez and Lazzeri all have their share:

Theorem 3.2. [14,5,9]. *Let M denote the Milnor fibre associated to a polynomial $f = x_1^{l_1+1} + \cdots + x_n^{l_n+1}$ of Brieskorn–Pham type stabilised to dimension $N \equiv 2(4)$. Then $H_N(M)$ has a geometrically distinguished basis $\{v_i\}$ indexed by the lexicographically ordered set of finite sequences of length n*

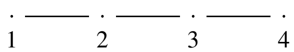
$$I = \{i_1 i_2 \cdots i_n \mid 1 \leqslant i_\nu \leqslant l_\nu \text{ for } 1 \leqslant \nu \leqslant n\},$$

and the intersection product is determined by

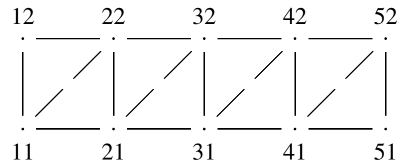
$$\langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } |i_\nu - j_\nu| > 1 \text{ for some } \nu, \\ 0 & \text{if } (i_\nu - j_\nu)(i_{\nu'} - j_{\nu'}) < 0 \text{ for some } \nu, \nu', \\ -(-1)^{\sum i_\nu - j_\nu} & \text{else.} \end{cases} \tag{1}$$

It is customary to encode the given datum $v_i, \langle v_i, v_j \rangle$ in a Dynkin diagram, a graph, of which the vertices are labelled in bijection to indices $i \in I$, with edges in bijection to (i, j) such that $\langle v_i, v_j \rangle = 1$ and broken edges in bijection to (i, j) such that $\langle v_i, v_j \rangle = -1$.

Example 1. In case $n = 1$ we draw the Dynkin diagram for $f(x) = x^5$ (of type A_4) and in case $n = 2$ we reproduce the diagram for $f(x) = x_1^6 + x_2^3$ (parabolic of type J_{10}, cf. [1]), which has been given by Gabrielov [5, Example 6]



A_4



J_{10}

4. Results

To state our results we use the band generators $\sigma_{i,j}$ of the braid group, cf. [2], which are conjugates of the standard (Artin-)generators $\sigma_i = \sigma_{i,i+1}$ and inductively related by the formula $\sigma_{i,j+1} = \sigma_j \sigma_{i,j} \sigma_j^{-1}$ for $j > i$:

Theorem 4.1. *Let f be a Brieskorn–Pham polynomial $f = x_1^{l_1+1} + \cdots + x_n^{l_n+1}$.*
Then the braid monodromy group is generated by the set of braids

$$\left\{\sigma_{i,j}^2 \mid \langle v_i, v_j \rangle = 0\right\} \cup \left\{\sigma_{i,j}^3 \mid \langle v_i, v_j \rangle \neq 0\right\} \cup \left\{\sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2} \mid \langle v_i, v_j \rangle \langle v_i, v_k \rangle \langle v_j, v_k \rangle \neq 0, i < j < k\right\}.$$

This result is obviously known in case $n = 1$, the case of type A singularities. The first part of the proof in [11] then establishes the claim in case $n = 2$, which is also a cornerstone in the second part to make induction work from $n = 2$ onwards. In fact an inductive argument starting from another known case like D_n or E_n type singularities should be possible with only minor adjustments in the iteration step, thus establishing a (weak) Thom–Sebastiani principle, cf. [15]: The braid monodromy group of $f(x) + g(y)$ with g of Brieskorn–Pham type is determined by that of f and that of g.

To use the theorem on the fundamental group we employ the argument of Zariski and van Kampen, cf. [10]. It relies on the choice of a geometric basis – or good ordered system [12] – which is a customary tool to identify the action of the group of isotopy classes of diffeomorphisms on the fundamental group of a fibre with the right Artin action of the abstract braid group on the free generators $t_1, \ldots, t_\mu$ given by $(t_i)\sigma_j = t_i$ if $j \neq i, i+1$, $(t_j)\sigma_j = t_j t_{j+1} t_j^{-1}$ and $(t_{j+1})\sigma_j = t_j$.

Theorem 4.2. *Given a hypersurface germ with braid monodromy group generated by braids $\{\beta_s\}$ in $\mathbf{Br}_n$. Then the fundamental group of the complement is presented as*

$$\left\langle t_1, \ldots, t_n \mid t_i^{-1}(t_i)\beta_s, 1 \leqslant i \leqslant n, \text{ for all } \beta_s \right\rangle.$$

In the case that a braid β_s is a conjugate of a power of σ_1, it is easy to see, that the relations $t_i^{-1}(t_i)\beta_s$ are the consequence of a single relation $t_{i_s}^{-1}(t_{i_s})\beta_s$, where i_s is the label of any of the both strands actually twisted by β_s. Hence with the generators of the braid monodromy group given by the theorem, the number of relations drops considerably and a straightforward computation yields:

Theorem 4.3. *Let f be a Brieskorn–Pham polynomial $f = x_1^{l_1+1} + \cdots + x_n^{l_n+1}$.*

Then the fundamental group of the discriminant complement in a versal unfolding of f is presentable with generators t_i in bijection to vanishing cycles v_i if $\{v_i, i \in I\}$ is a basis of the Milnor lattice of f as above

$$\left\langle t_i, i \in I \;\middle|\; \begin{array}{ll} t_i t_j = t_j t_i & \text{if } \langle v_i, v_j\rangle = 0, \\ t_i t_j t_i = t_j t_i t_j & \text{if } \langle v_i, v_j\rangle \neq 0, \\ t_i t_j t_k t_i = t_j t_k t_i t_j & \text{if } \langle v_i, v_j\rangle\langle v_i, v_k\rangle\langle v_j, v_k\rangle \neq 0, i < j < k. \end{array} \right\rangle$$

The appealing aspects of this assertion are

(i) that it obviously generalises the corresponding claim for simple hypersurface singularities with respect to the tree intersection diagrams for their Milnor lattices, where the last case is void,
(ii) and that it rises immediate hopes that its validity extends to even more general singularities.

However, although we have some support for this hope, one has to be very cautious, since the 'correct' Dynkin diagram has to be found, which seems not an easy task for a general singularity.

Remarks. (1) In the terminology of F. Napolitano [13], who extended the Enriques–Zariski ideas and defined pseudo-homology groups $\mathcal{H}_i$, we have given a finite presentation of $\mathcal{H}_0(\mathcal{D}) \cong \pi_1(\mathbf{C}^\mu - \mathcal{D})$ but we think that to approach the higher groups one needs other geometric ideas.

(2) Givental and Shekhtman, cf. [7,6], introduced a refined monodromy map with domain $\pi_1(\mathbf{C}^\mu - \mathcal{D})$ and values in a generalised Hecke algebra. It may be worthwhile to determine the kernel and use our explicit presentation.

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