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Algebraic Geometry

# Semi-universal deformation spaces of some simple elliptic singularities

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**Abstract**

In this Note, we deal with the simple elliptic singularities of type $\tilde{D}_5$. By using the Lie algebra $sl(2,\mathbb{C})\oplus sl(2,\mathbb{C})$, we construct semi-universal deformation spaces of these singularities. ***To cite this article: K. Nakamoto, M. Tosun, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***


**Résumé**

**Déformation semi-universelle des singularités elliptiques simples.** Dans cette Note, nous traitons les singularités elliptiques simples du type $\tilde{D}_5$. En utilisant l'algèbre de Lie $sl(2,\mathbb{C})\oplus sl(2,\mathbb{C})$, nous construisons des espaces de déformation semi-universelle de ces singularités. ***Pour citer cet article : K. Nakamoto, M. Tosun, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***


## 1. Introduction

Simple elliptic singularities of normal surfaces were defined by Saito in [5], and several special types were named as $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ and $\tilde{D}_5$. Beyond Grothendieck–Brieskorn theory on the relation between simple singularities of surfaces and simple Lie algebras (see [1,7]), many mathematicians tried to discover some similar relations between simple elliptic singularities and Lie algebras or related objects ([4,6] and so on).

Here we construct the simple elliptic singularities of type $\tilde{D}_5$ and their semi-universal deformation spaces by using $sl(2,\mathbb{C})\oplus sl(2,\mathbb{C})$. This construction contrasts with the one of Helmke and Slodowy [2] who used a loop group, i.e. an infinite dimensional object.

## 2. Nilpotent variety and its singularities

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$. The nilpotent variety $\mathcal{N}(\mathfrak{g})$ of $\mathfrak{g}$ is defined as $\mathcal{N}(\mathfrak{g}) := \{x \in \mathfrak{g} \mid \mathrm{Ad}(x) : \mathfrak{g} \to \mathfrak{g} \text{ is nilpotent}\}$.

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In the case of $\mathfrak{g} = sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})$ the nilpotent variety $\mathcal{N} := \mathcal{N}(\mathfrak{g})$ is

$$\mathcal{N} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| a^2 + bc = 0 \right\} \times \left\{ \begin{pmatrix} d & e \\ f & -d \end{pmatrix} \middle| d^2 + ef = 0 \right\}.$$

Let us consider a 4-dimensional affine subspace $\mathcal{S}$ of $\mathfrak{g}$ passing through the origin $0 = (O, O) \in \mathcal{N}$. We say that $\mathcal{S}$ is a *generic slice at* 0 if the intersection $\mathcal{N} \cap \mathcal{S}$ is an isolated surface singularity at 0. In the sequel, we assume that $\mathcal{S}$ is a generic slice.

**Remark 2.1.** More precisely, we can define the genericity of slices in the following way: Let us fix an isomorphism $\mathcal{S} \cong \mathbb{C}^4$. A quadratic equation in $S$ can be written as $(x, y, z, t)^t A(x, y, z, t) = 0$ with $A$ a symmetric $4 \times 4$ matrix. Hence, for a slice $\mathcal{S}$, we obtain two quadratic equations $f = (a^2 + bc)|_{\mathcal{S}}$ and $g = (d^2 + ef)|_{\mathcal{S}}$ in $\mathbb{C}^4$ which are expressed by symmetric matrices $A$ and $B$, respectively. We say that $\mathcal{S}$ is generic if the discriminant of the polynomial $\det(tA + B)$ is non-zero (see [3]).

**Proposition 2.2.** *With the preceding notation, the surface singularity* $(X, 0) := (\mathcal{N} \cap \mathcal{S}, 0)$ *is a simple elliptic singularity of type* $\tilde{D}_5$.

**Proof.** Let $\tilde{\mathcal{S}}$ be the blowing up of $\mathcal{S} \cong \mathbb{C}^4$ at 0. By taking the strict transform $\tilde{X}$ of $X$, we have

$$\begin{array}{ccccc} \mathbb{P}^3 & \subset & \tilde{\mathcal{S}} & \to & \mathcal{S} \\ \cup & & \cup & & \cup \\ E & \subset & \tilde{X} & \to & X, \end{array}$$

where $E$ is the exceptional curve. Since $X$ is defined by two quadratic equations in $S$, the exceptional curve $E$ will be defined by two generic quadratic equations in $\mathbb{P}^3$. Hence $E$ is an elliptic curve and $E^2 = -4$. Therefore $(X, 0)$ is a simple elliptic singularity of type $\tilde{D}_5$ (see [5]). □

## 3. Semi-universal deformations

It is well known that $(X, 0)$ has a semi-universal deformation space, the base space is non-singular of dimension $\dim T^1$ and, $\dim T^2 = 0$ [8].

**Proposition 3.1.** *For each* $\tilde{D}_5$*-singularity,* $\dim T^1 = 7$.

**Proof.** Any singularities of type $\tilde{D}_5$ are given by the equations $f = x_1^2 + x_2^2 + \lambda x_3 x_4 = 0$ and $g = x_1 x_2 + x_3^2 + x_4^2 = 0$ in $\mathbb{C}^4$ with some $\lambda \in \mathbb{C} \setminus \{0, \pm 4\}$. Then an easy calculation gives $\dim T^1 = 7$. □

To construct semi-universal deformations of $\tilde{D}_5$-singularity, we will first restrict ourselves to the special case where $\mathcal{S}_0 := \{c = d + e,\ f = a + b\} \subset \mathfrak{g}$ and denote $(X_0, 0) := (\mathcal{N} \cap \mathcal{S}_0, 0)$. Consider a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ defined as

$$\mathfrak{h} := \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} d & 0 \\ 0 & -d \end{pmatrix} \right\}.$$

The adjoint quotient $\mathfrak{g} \to \mathfrak{h}/W$ can be regarded as

$$\begin{array}{rccc} \chi : & \mathfrak{g} & \to & \mathfrak{h}/W \cong \mathbb{C}^2, \\ & \left( \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} d & e \\ f & -d \end{pmatrix} \right) & \mapsto & \left(-a^2 - bc, -d^2 - ef\right), \end{array}$$

where $W$ is the Weyl group of $\mathfrak{g}$ which is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Let us deform the adjoint quotient $\chi$ by $(\alpha, \beta) \in \mathbb{C}^2$ as

$$\begin{array}{rccc} f_{(\alpha,\beta)} : & \mathfrak{g} & \to & \mathfrak{h}/W \cong \mathbb{C}^2, \\ & \left( \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} d & e \\ f & -d \end{pmatrix} \right) & \mapsto & \left(-a^2 - bc - \alpha e, -d^2 - ef - \beta b\right). \end{array}$$

When $(\alpha, \beta) = (0, 0)$, we have $f_{(0,0)} = \chi$. And, let us deform the slice $\mathcal{S}_0$ by $(\gamma, \delta, \varepsilon) \in \mathbb{C}^3$ as

$$\mathcal{S}_{(\gamma,\delta,\varepsilon)} := \{c = d + e + \gamma,\ f = a + b + \delta e + \varepsilon\}.$$

For $(\gamma, \delta, \varepsilon) = (0, 0, 0)$, we have $\mathcal{S}_{(0,0,0)} = \mathcal{S}_0$.

**Theorem 3.2.** *With the preceding notation, consider*

$$S := \mathbb{C}^2 \times \mathbb{C}^3 \times \mathfrak{h}/W = \{(\alpha, \beta) \in \mathbb{C}^2\} \times \{(\gamma, \delta, \varepsilon) \in \mathbb{C}^3\} \times \{(\lambda, \mu) \in \mathfrak{h}/W\}.$$

*Let $\mathcal{X}$ be the family of surfaces on $S$ defined as*

$$\mathcal{X} := \{(X, \alpha, \beta, \gamma, \delta, \epsilon, \lambda, \mu) \in \mathfrak{g} \times S \mid f_{(\alpha,\beta)}(X) = (\lambda, \mu),\ X \in \mathcal{S}_{(\gamma,\delta,\varepsilon)}\}.$$

*Set $o := (0, 0, 0, 0, 0, 0, 0) \in S$ and $q := (O, o) \in \mathcal{X}$. Then the morphism of germs $(\mathcal{X}, q) \to (S, o)$ gives a semi-universal deformation of $(X_0, 0)$.*

**Proof.** The coordinate ring $\mathcal{O}_{X_0}$ of $(X_0, 0)$ is isomorphic to $\mathbb{C}\{a, b, d, e\}/(g_1, g_2)$, where $g_1 = a^2 + bd + be$ and $g_2 = d^2 + ae + be$. The $\mathbb{C}$-vector space $T^1 = \mathcal{O}^2_{X_0}/M$, where $M$ is the $\mathcal{O}_{X_0}$-submodule of $\mathcal{O}^2_{X_0}$ generated by the 4 vectors: $(\frac{\partial g_1}{\partial a}, \frac{\partial g_2}{\partial a})$, $(\frac{\partial g_1}{\partial b}, \frac{\partial g_2}{\partial b})$, $(\frac{\partial g_1}{\partial d}, \frac{\partial g_2}{\partial d})$ and $(\frac{\partial g_1}{\partial e}, \frac{\partial g_2}{\partial e})$.

Since $f_{(\alpha,\beta)}(X) = (-a^2 - bc - \alpha e, -d^2 - ef - \beta b)$ and $c = d + e + \gamma$, $f = a + b + \delta e + \varepsilon$ for $X \in \mathcal{S}_{(\gamma,\delta,\varepsilon)}$, the family $\mathcal{X}$ is defined by

$$f_1 := a^2 + bd + be + \gamma b + \alpha e + \lambda = 0$$
$$f_2 := d^2 + ae + be + \delta e^2 + \varepsilon e + \beta b + \mu = 0.$$

Note that we have

$$(f_1, f_2) = (g_1, g_2) + \alpha(e, 0) + \beta(0, b) + \gamma(b, 0) + \delta(0, e^2) + \varepsilon(0, e) + \lambda(1, 0) + \mu(0, 1)$$

and the 7 vectors appeared over $(e, 0)$, $(0, b)$, $(b, 0)$, $(0, e^2)$, $(0, e)$, $(1, 0)$, and $(0, 1)$ form a basis for $T^1$. Hence the family $(\mathcal{X}, q) \to (S, o)$ induces an isomorphism from $S$ to $T^1$. Therefore $(\mathcal{X}, q) \to (S, o)$ is isomorphic to a semi-universal deformation of $(X_0, 0)$. □

Now we want to construct semi-universal deformation spaces for a general transversal slice. For this, consider the space $\mathrm{Aff}(\mathfrak{g}, 4)$ of all 4-dimensional affine subspaces of $\mathfrak{g}$. Since any 4-dimensional affine subspace of $\mathfrak{g}$ can be described by two linear equations, $\mathrm{Aff}(\mathfrak{g}, 4)$ is embedded in the Grassmann variety $\mathrm{Grass}(\dim \mathfrak{g} + 1, 2) = \mathrm{Grass}(7, 2)$. The space of all 4-dimensional linear subspaces $\mathrm{Grass}(\mathfrak{g}, 4)$ of $\mathfrak{g}$ is a closed subvariety of $\mathrm{Aff}(\mathfrak{g}, 4)$.

By Proposition 2.2, $(\mathcal{N} \cap \mathcal{S}, 0)$ gives us a $\tilde{D}_5$-singularity for a general $\mathcal{S}$ in $\mathrm{Grass}(\mathfrak{g}, 4)$. Then we obtain:

**Theorem 3.3.** *Let $\mathcal{S}$ be a general element of* $\mathrm{Grass}(\mathfrak{g}, 4)$. *Let $\mathcal{S}_*$ be a 'general' 3-dimensional subvariety passing through $\mathcal{S}$ of* $\mathrm{Aff}(\mathfrak{g}, 4)$. *Set*

$$S := \mathbb{C}^2 \times \mathcal{S}_* \times \mathfrak{h}/W = \{(\alpha, \beta) \in \mathbb{C}^2\} \times \{\mathcal{T} \in \mathcal{S}_*\} \times \{(\lambda, \mu) \in \mathfrak{h}/W \cong \mathbb{C}^2\}$$

*and*

$$\mathcal{X} := \{(X, \alpha, \beta, \mathcal{T}, \lambda, \mu) \in \mathfrak{g} \times S \mid f_{(\alpha,\beta)}(X) = (\lambda, \mu),\ X \in \mathcal{T}\}.$$

*Then the morphism of germs $(\mathcal{X}, q) \to (S, o)$ gives us a semi-universal deformation of $(\mathcal{N} \cap \mathcal{S}, 0)$, where $o = (0, 0, \mathcal{S}, 0, 0)$ and $q = (0, o)$.*

**Proof.** The condition that 7 vectors are linearly independent in $T^1 = \mathcal{O}^2_{\mathcal{N}\cap\mathcal{S}}/M$ is open for $\mathcal{S} \in \mathrm{Grass}(\mathfrak{g}, 4)$. Hence the condition that a given family becomes a semi-universal deformation is also open. Then we can choose a suitable 3-dimensional subvariety of $\mathrm{Aff}(\mathfrak{g}, 4)$, which implies the meaning of the word 'general'. □

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