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Algebraic Geometry

Multiplicity of complex hypersurface singularities, Rouché satellites and Zariski's problem ☆

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Abstract

Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be reduced germs of holomorphic functions. We show that f and g have the same multiplicity at 0, if and only if, there exist reduced germs f' and g' analytically equivalent to f and g, respectively, such that f' and g' satisfy a Rouché type inequality with respect to a generic 'small' circle around 0. As an application, we give a reformulation of Zariski's multiplicity question and a partial positive answer to it. ***To cite this article: C. Eyral, E. Gasparim, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Multiplicité des singularités d'hypersurfaces complexes, satellites de Rouché et problème de Zariski. Soient $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ des germes de fonctions holomorphes réduits. Nous montrons que f et g ont la même multiplicité en 0 si et seulement s'il existe des germes réduits f' et g' analytiquement équivalents à f et g, respectivement, tels que f' et g' satisfassent une inégalité du type de Rouché par rapport à un 'petit' cercle générique autour de 0. Comme application, nous donnons une reformulation de la question de Zariski sur la multiplicité et une réponse partielle positive à celle-ci. ***Pour citer cet article : C. Eyral, E. Gasparim, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

1. Introduction

Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be reduced germs (at the origin) of holomorphic functions, with $n \geqslant 2$, V_f, V_g the corresponding germs of hypersurfaces in $\mathbb{C}^n$, and ν_f, ν_g the multiplicities at 0 of V_f, V_g respectively. By the *multiplicity* ν_f we mean the number of points of intersection, near 0, of V_f with a generic (complex) line in $\mathbb{C}^n$ passing arbitrarily close to 0 but not through 0. As we are assuming that f is reduced, ν_f is also the *order* of f at 0, that is, the lowest

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degree in the power series expansion of f at 0. We denote by $C(V_f)$, $C(V_g)$ the tangent cones at 0 of V_f, V_g, that is, the zero sets of the initial polynomials of f and g respectively (cf. [13]).

In Section 2, we prove that $\nu_f = \nu_g$, if and only if, there exist reduced germs f' and g' analytically equivalent to f and g, respectively, such that $|f'(z) - g'(z)| < |f'(z)|$, for all $z \in \dot{D}$, where $\dot{D}$ is the boundary of a generic 'small' disc around 0 (Theorem 2.6). We call such an inequality a *Rouché inequality* and we say that g' is a *Rouché satellite* of f'.

In Section 3, we apply this result to Zariski's multiplicity question. In particular, we show that the answer to Zariski's question is *yes*, if and only if, for any two topologically equivalent reduced germs f and g there exist reduced germs f' and g' analytically equivalent to f and g, respectively, such that g' is a Rouché satellite of f' (Theorem 3.6). In addition, we answer positively Zariski's question in the special case of 'small' homeomorphisms for Newton nondegenerate isolated singularities (Corollary 3.3) and one-parameter families of isolated singularities (Corollary 3.5).

2. Multiplicity and Rouché satellites

Let L be a line through 0 in $\mathbb{C}^n$ not contained in $C(V_f) \cup C(V_g)$ (equivalently, $L \cap (C(V_f) \cup C(V_g)) = \{0\}$). Then ν_f (respectively ν_g) is the order at 0 of $f_{|L}$ (respectively $g_{|L}$), and 0 is an isolated point of $L \cap V_f$ and $L \cap V_g$ (cf. [2]). In particular, there exists a closed disc $D \subseteq L$ around 0 such that, for any closed disc $D' \subseteq D$ around 0, $D' \cap (V_f \cup V_g) = \{0\}$. We shall call such a disc D a *good* disc for f and for g.

Definition 2.1. We say that g is a *Rouché satellite* of f if there exists a good disc D (for f and for g) such that f and g satisfy a *Rouché inequality* with respect to the boundary $\dot{D}$ of D, that is,

$$\left|f(z) - g(z)\right| < \left|f(z)\right|$$

for all $z \in \dot{D}$.

Theorem 2.2. *If g is a Rouché satellite of f, then $\nu_g = \nu_f$.*

Proof. Let $D \subseteq L$ be a good disc for f and for g (for some line L through 0 not contained in $C(V_f) \cup C(V_g)$) such that $|f_{|L}(z) - g_{|L}(z)| < |f_{|L}(z)|$ for all $z \in \dot{D}$. By Rouché theorem (cf. e.g. [7, Chapter VI, Theorem 1.6]), $f_{|L}$ and $g_{|L}$ have the same number of zeros, counted with their multiplicities, in the interior of D. Thus, since $f_{|L}$ and $g_{|L}$ vanish only at 0 on D, the orders at 0 of $f_{|L}$ and $g_{|L}$ are equal. In other words, $\nu_f = \nu_g$. □

Example 2.3. Consider the germs $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ defined by

$$f(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_3^3 + z_1^3 + z_2^4 \quad \text{and} \quad g(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_3^3 + z_1^4 + z_2^6.$$

Then g is a Rouché satellite of f. Indeed, set $L = \{(z_1, 0, z_3) \in \mathbb{C}^3 \mid z_1 = z_3\}$; then

$$V_f \cap L = \left\{(0,0,0), \left(-\frac{1}{2}, 0, -\frac{1}{2}\right)\right\} \quad \text{and} \quad V_g \cap L = \left\{(0,0,0), (a, 0, a), (\bar{a}, 0, \bar{a})\right\},$$

where $a = (-1 - \mathrm{i}\sqrt{3})/2$ and $\bar{a}$ is the complex conjugate of a. So, the disc $D \subseteq L$ of radius $1/4$ is good for f and for g, and, for all $z \in \dot{D}$,

$$\left|f(z) - g(z)\right| \leqslant \frac{5}{4^4} < \frac{2}{4^3} \leqslant \left|f(z)\right|.$$

Hence g is a Rouché satellite of f. In fact, here, f is also a Rouché satellite of g. Indeed, for all $z \in \dot{D}$, we have

$$\left|f(z) - g(z)\right| \leqslant \frac{5}{4^4} < \frac{11}{4^4} \leqslant \left|g(z)\right|.$$

Of course, in general, g may be a Rouché satellite of f without f being a Rouché satellite of g. For example, take $g = f/2$. Also, it is not difficult to construct f and g such that $\nu_f = \nu_g$ but neither g is a Rouché satellite of f nor f a Rouché satellite of g. Take for example $g = -f$. Nevertheless, such an unpleasant situation is resolved by Theorem 2.5 below.

Definition 2.4. If there exists a germ of homeomorphism $\varphi:(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ such that:

(1) $\varphi(V_g)=V_f$ then f and g are called *topologically equivalent* (denoted $f\sim_t g$);
(2) $\varphi(V_g)=V_f$ and φ is an analytic isomorphism, then f and g are called *analytically equivalent* (denoted $f\sim_a g$);
(3) $g=f\circ\varphi$ then f and g are called *topologically right equivalent* (denoted $f\sim_{tr} g$).

Note that the definition makes sense only for *reduced* germs. In the special case of an isolated singularity, the hypothesis '$n\geqslant 2$' automatically implies that the germ is reduced. Note also that (2) ⇒ (1) and (3) ⇒ (1).

Theorem 2.2 has the weak following converse:

Theorem 2.5. *If $\nu_f=\nu_g$, then there exist reduced germs $f'\sim_a f$ and $g'\sim_a g$ such that g' is a Rouché satellite of f'.*

Proof. By an analytic change of coordinates, one can assume that the z_n-axis, Oz_n, is not contained in the tangent cones $C(V_f)$, $C(V_g)$, so that $f(0,\dots,0,z_n)\neq 0$ and $g(0,\dots,0,z_n)\neq 0$, for any $z_n\neq 0$ close enough to 0. By the Weierstrass preparation theorem, for z near 0, the germ $f(z)$ can be represented as a product $f(z)=f'(z)\,f''(z)$, where $f''(z)$ is a germ of holomorphic function which does not vanish around 0 and where $f'(z)$ is of the form

$$f'(z_1,\dots,z_n)=z_n^{\nu_f}+z_n^{\nu_f-1}f_1(z_1,\dots,z_{n-1})+\cdots+f_{\nu_f}(z_1,\dots,z_{n-1}),$$

with, for $1\leqslant i\leqslant\nu_f$, $f_i\in\mathbb{C}\{z_1,\dots,z_{n-1}\}$, $f_i(0)=0$ and the order of f_i at 0 is $\geqslant i$. Similarly $g(z)=g'(z)\,g''(z)$, with $g''(z)\neq 0$ for all z near 0, and

$$g'(z_1,\dots,z_n)=z_n^{\nu_g}+z_n^{\nu_g-1}g_1(z_1,\dots,z_{n-1})+\cdots+g_{\nu_g}(z_1,\dots,z_{n-1}),$$

with, for $1\leqslant i\leqslant\nu_g$, $g_i\in\mathbb{C}\{z_1,\dots,z_{n-1}\}$, $g_i(0)=0$ and the order of g_i at 0 is $\geqslant i$. Clearly f' and g' are reduced, and, since $V_f=V_{f'}$ and $V_g=V_{g'}$, $f'\sim_a f$ and $g'\sim_a g$. On the other hand, since $\nu_f=\nu_g$, $f'_{|Oz_n}=g'_{|Oz_n}$. But for any disc $D\subseteq Oz_n$ around 0 (in particular for any good disc in Oz_n for f' and g'), $|f'(z)|=r^{\nu_f}\neq 0$ for all $z\in\dot{D}$, where r is the radius of D. □

Since the multiplicity is an invariant of the (embedded) reduced analytic type, we can summarize Theorems 2.2 and 2.5 as follows:

Theorem 2.6. *The multiplicities ν_f and ν_g are the same, if and only if, there exist reduced germs $f'\sim_a f$ and $g'\sim_a g$ such that g' is a Rouché satellite of f'.*

3. Applications to Zariski's multiplicity question

In [14], Zariski posed the following question: *if $f\sim_t g$, then is it true that $\nu_f=\nu_g$?* The question is, in general, still unsettled (even for hypersurfaces with isolated singularities). The answer is, nevertheless, known to be *yes* in several special cases the list of which can be found in the recent first author's survey article [3]. In particular, Ephraim [2] proved that multiplicity is preserved by ambient C^1-diffeomorphisms; his paper inspired some of our proofs. In this section, we give a partial positive answer to Zariski's question in the special case of 'small' homeomorphisms for Newton nondegenerate isolated singularities and one-parameter families of isolated singularities. In addition, we give an equivalent reformulation of Zariski's question in terms of Rouché satellites.

We start with the following result which asserts that if f and g are topologically right equivalent via a sufficiently 'small' homeomorphism, then they have the same multiplicity. More precisely suppose $f\sim_{tr} g$. Then there are representatives $\mathrm{f}:U\to\mathbb{C}$ and $\mathrm{g}:U'\subseteq U\to\mathbb{C}$ of the germs f and g respectively and a homeomorphism $\varphi:U'\to\varphi(U')\subseteq U$ such that $\varphi(0)=0$ and $\mathrm{g}=\mathrm{f}\circ\varphi$. Since f is uniformly continuous on a compact small ball $B_r\subseteq U'$ around 0, there exists $\eta>0$ such that, for any $z,w\in B_r$,

$$|z-w|<\eta\ \Rightarrow\ \left|\mathrm{f}(z)-\mathrm{f}(w)\right|<\inf_{u\in\dot{D}_\varrho}\left|\mathrm{f}(u)\right|,$$

where D_ϱ is a good disc at 0 for f and for $\mathrm{g}=\mathrm{f}\circ\varphi$ with radius $\varrho\leqslant r/2$.

Definition 3.1. We will say that the homeomorphism $\varphi : U' \to \varphi(U') \subseteq U$ is f-*small* if there exists a triple (r, ϱ, η) as above such that, for all $z \in B_r$, $|z - \varphi(z)| < \inf\{\eta, \varrho\}$.

Theorem 3.2. *With the above hypotheses and notation, if the homeomorphism $\varphi : U' \to \varphi(U') \subseteq U$ is* f*-small, then $\nu_f = \nu_g$.*

Proof. By hypothesis, for all $z \in \dot{D}_\varrho$, $\varphi(z) \in B_r$ and $|\mathrm{f}(z) - \mathrm{f} \circ \varphi(z)| < \inf_{u \in \dot{D}_\varrho} |\mathrm{f}(u)| \leqslant |\mathrm{f}(z)|$. Therefore $\mathrm{g} = \mathrm{f} \circ \varphi$ is a Rouché satellite of f. Then, by Theorem 2.2, $\nu_f = \nu_g$. □

The interest in topologically right equivalent germs with regard to Zariski's question comes from the following. By theorems of King [4], Perron [10], Saeki [11] and Nishimura [8], if f has an *isolated* singularity at 0 and a nondegenerate Newton principal part, then the relation $f \sim_t g$ implies $f \sim_{tr} g$. On the other hand, by another theorem of King [5], for a one-parameter holomorphic family of *isolated* singularities $(f_s)_s$ in $\mathbb{C}^n$, with $n \neq 3$, if the relation $f_s \sim_t f_0$ holds for all s near 0, then so does $f_s \sim_{tr} f_0$. So, when considering isolated Newton nondegenerate singularities or *families* of isolated singularities, the Zariski problem refers immediately to right equivalent germs.

Corollary 3.3. *Assume that f has an isolated critical point at* 0 *and a nondegenerate Newton principal part, and suppose $g \sim_t f$. In this case, there are representatives $\mathrm{f} : U \to \mathbb{C}$ and $\mathrm{g} : U' \subseteq U \to \mathbb{C}$ of f and g respectively and a homeomorphism $\varphi : U' \to \varphi(U') \subseteq U$ such that $\varphi(0) = 0$ and $\mathrm{g} = \mathrm{f} \circ \varphi$. If φ is* f*-small, then $\nu_f = \nu_g$.*

Remark 3.4. If, in addition, f is *convenient* (cf. [6]), then the hypothesis of having an isolated singularity at 0 is automatically satisfied (cf. [9]).

Corollary 3.3 is complementary to the result of Abderrahmane and Saia–Tomazella concerning μ-constant *families* of convenient Newton nondegenerate (isolated) singularities (cf. [1] and [12]).

Corollary 3.5. *Let $(f_s)_s$ be a topologically constant (or μ-constant) one-parameter holomorphic family of isolated hypersurface singularities, with $n \neq 3$. In this case, for all s near* 0*, there are representatives $\mathrm{f}_0 : U_0 \to \mathbb{C}$ and $\mathrm{f}_s : U_s \subseteq U_0 \to \mathbb{C}$ of f_0 and f_s respectively and a homeomorphism $\varphi_s : U_s \to \varphi(U_s) \subseteq U_0$ such that $\varphi_s(0) = 0$ and $\mathrm{f}_s = \mathrm{f}_0 \circ \varphi_s$. If, for all s near* 0*, φ_s is f_0-small, then $(f_s)_s$ is equimultiple (i.e., for all s near* 0*, $\nu_{f_s} = \nu_{f_0}$).*

We conclude with the following nice consequence of Theorem 2.6 which is reformulation of Zariski's multiplicity question in terms of Rouché satellites:

Theorem 3.6. *The answer to Zariski's multiplicity question is yes, if and only if, the relation $f \sim_t g$ implies that there exist reduced germs $f' \sim_a f$ and $g' \sim_a g$ such that g' is a Rouché satellite of f'.*

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