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Mathematical Problems in Mechanics

Recovery of a displacement field on a surface from its linearized change of metric and change of curvature tensors

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Abstract

We establish that the components of the linearized change of metric and change of curvature tensors associated with a displacement field of a surface in $\mathbb{R}^3$ must satisfy compatibility conditions, which are the analogues 'on a surface' of the Saint Venant equations in three-dimensional elasticity.

We next show that, conversely, if two symmetric matrix fields of order two satisfy these compatibility conditions over a simply-connected surface $S \subset \mathbb{R}^3$, then they are the linearized change of metric and change of curvature tensors associated with a displacement field of the surface S. ***To cite this article: P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Reconstruction d'un champ de déplacements d'une surface à partir de ses tenseurs linéarisés de changement de métrique et de changement de courbure. On montre que les composantes des tenseurs linéarisés de changement de métrique et de changement de courbure associés à un champ de déplacements d'une surface de $\mathbb{R}^3$ doivent satisfaire certaines relations de compatibilité, qui sont les analogues « sur une surface » des relations de Saint Venant en élasticité tri-dimensionnelle.

On montre ensuite que, inversement, si deux champs de matrices symétriques d'ordre deux satisfont ces mêmes relations de compatibilité sur une surface $S \subset \mathbb{R}^3$ simplement connexe, alors ce sont les tenseurs linéarisés de changement de métrique et de changement de courbure d'un champ de déplacements de la surface S. ***Pour citer cet article : P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***
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Version française abrégée

Les notations et définitions utilisées ici sont précisées dans la version anglaise. Le résultat principal de cette Note est le suivant (Théorème 3.1 de la version anglaise) : Soit ω un domaine simplement connexe de $\mathbb{R}^2$ et soit $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{R}^3)$

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une immersion. Soit par ailleurs $(\gamma_{\alpha\beta}) \in L^2(\omega;\mathbb{S}^2)$ et $(\rho_{\alpha\beta}) \in L^2(\omega;\mathbb{S}^2)$ deux champs de matrices symétriques qui satisfont au sens des distributions les *équations de Saint Venant sur la surface* $S = \boldsymbol{\theta}(\bar{\omega})$, à savoir :

$$\gamma_{\sigma\alpha|\beta\tau} + \gamma_{\tau\beta|\alpha\sigma} - \gamma_{\tau\alpha|\beta\sigma} - \gamma_{\sigma\beta|\alpha\tau} + R^{\nu}_{\cdot\alpha\sigma\tau}\gamma_{\beta\nu} - R^{\nu}_{\cdot\beta\sigma\tau}\gamma_{\alpha\nu} = b_{\tau\alpha}\rho_{\sigma\beta} + b_{\sigma\beta}\rho_{\tau\alpha} - b_{\sigma\alpha}\rho_{\tau\beta} - b_{\tau\beta}\rho_{\sigma\alpha},$$
$$\rho_{\sigma\alpha|\tau} - \rho_{\tau\alpha|\sigma} = b^{\nu}_{\sigma}(\gamma_{\alpha\nu|\tau} + \gamma_{\tau\nu|\alpha} - \gamma_{\tau\alpha|\nu}) - b^{\nu}_{\tau}(\gamma_{\alpha\nu|\sigma} + \gamma_{\sigma\nu|\alpha} - \gamma_{\sigma\alpha|\nu}).$$

Alors il existe un champ de vecteurs $\boldsymbol{\eta} = \eta_i \mathbf{a}^i \in H^1(\omega;\mathbb{R}^3)$ tel que

$$\gamma_{\alpha\beta} = \frac{1}{2}(\partial_\alpha\boldsymbol{\eta}\cdot\mathbf{a}_\beta + \mathbf{a}_\alpha\cdot\partial_\beta\boldsymbol{\eta}) \quad \text{dans } L^2(\omega),$$
$$\rho_{\alpha\beta} = (\partial_{\alpha\beta}\boldsymbol{\eta} - \Gamma^{\nu}_{\alpha\beta}\partial_\nu\boldsymbol{\eta})\cdot\mathbf{a}_3 \quad \text{dans } H^{-1}(\omega).$$

Autrement dit, les champs $(\gamma_{\alpha\beta})$ et $(\rho_{\alpha\beta})$ sont les *champs de tenseurs linéarisés de changement de métrique et de changement de courbure associés au champ de déplacements* $\boldsymbol{\eta}$ *de la surface* S. La démonstration de ce résultat est constructive, dans le sens qu'elle fournit un algorithme explicite de construction du champ $\boldsymbol{\eta}$ à partir des champs $(\gamma_{\alpha\beta})$ et $(\rho_{\alpha\beta})$.

Ce résultat peut être vu comme la « version infinitésimale » de la construction d'une surface à partir de ses deux formes fondamentales, car les équations de Saint Venant sur une surface (qui constituent une condition nécessaire si le champ $\boldsymbol{\eta}$ est donné ; cf. Théorème 2.1 de la version anglaise), ne sont autres que la partie « du premier ordre en ε » des équations de Gauss et Codazzi–Mainardi associées à l'immersion $(\boldsymbol{\theta} + \varepsilon\boldsymbol{\eta})$ pour $|\varepsilon|$ suffisament petit (Théorème 4.3 de la version anglaise).

On trouvera dans [3] les démonstrations détaillées de ces résultats, ainsi que divers compléments, notamment l'application de ces résultats à la justification de la théorie « intrinsèque » des coques (cf. [2]).

1. Notations and preliminaries

Latin indices and exponents vary in the set $\{1,2,3\}$, Greek indices and exponents vary in the set $\{1,2\}$, and the summation convention with respect to repeated indices and exponents is systematically used in conjunction with this rule.

The Euclidean inner product and the vector product of $\boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^3$ and the Euclidean norm of $\boldsymbol{u} \in \mathbb{R}^3$ are respectively denoted by $\boldsymbol{u}\cdot\boldsymbol{v}$, $\boldsymbol{u}\wedge\boldsymbol{v}$, and $|\boldsymbol{u}|$. The notation $(t_{\alpha\beta})$ designates the matrix in $\mathbb{M}^2$ with $t_{\alpha\beta}$ as its elements, the first index α being the row index. The symbols $\mathbb{S}^2$, $\mathbb{S}^2_>$, and $\mathbb{A}^2$ respectively designate the sets of all symmetric, positive-definite symmetric, and antisymmetric, matrices of order two.

Let ω be an open subset of $\mathbb{R}^2$. The coordinates of a point $y \in \omega$ are denoted y_α and we let $\partial_\alpha := \partial/\partial x_\alpha$ and $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$.

A *domain* in $\mathbb{R}^2$ is a bounded and connected open set with a Lipschitz-continuous boundary, the set ω being locally on the same side of its boundary. In what follows, we will need the following generalization of Poincaré's theorem (which is classically proved only for continuously differentiable functions), which is due to Ciarlet and Ciarlet, Jr. [1]:

Theorem 1.1. *Let ω be a simply-connected domain of $\mathbb{R}^2$. Let $h_\alpha \in H^{-1}(\omega)$ be distributions that satisfy*

$$\partial_\beta h_\alpha = \partial_\alpha h_\beta \quad \text{in } H^{-2}(\omega).$$

Then there exists a function $p \in L^2(\omega)$, unique up to an additive constant, such that

$$h_\alpha = \partial_\alpha p \quad \text{in } H^{-1}(\omega).$$

We now list the various definitions and properties from the differential geometry of surfaces in $\mathbb{R}^3$ that are needed in the detailed proofs of the theorems stated in this Note.

Let ω be a bounded open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega};\mathbb{R}^3)$ be an immersion. Then the image $S := \boldsymbol{\theta}(\bar{\omega})$ is a surface immersed in $\mathbb{R}^3$. For each $y \in \bar{\omega}$, the vectors $\mathbf{a}_\alpha(y) := \partial_\alpha\boldsymbol{\theta}(y)$ form a basis in the tangent space to the surface $\boldsymbol{\theta}(\omega)$ at the point $\boldsymbol{\theta}(y)$. The tangent vector fields $\mathbf{a}^\beta$, defined by $\mathbf{a}_\alpha(y)\cdot\mathbf{a}^\beta(y) = \delta^\beta_\alpha$ for all $y \in \bar{\omega}$, form the dual bases. A unit normal vector to S at $\boldsymbol{\theta}(y)$ is defined by

$$\mathbf{a}_3(y) = \mathbf{a}^3(y) := \frac{\mathbf{a}_1(y)\wedge\mathbf{a}_2(y)}{|\mathbf{a}_1(y)\wedge\mathbf{a}_2(y)|}.$$

The covariant components of the first fundamental form of S are defined by $a_{\alpha\beta}(y) = \mathbf{a}_\alpha(y) \cdot \mathbf{a}_\beta(y)$ for all $y \in \bar{\omega}$, and the contravariant components of the same form are defined by $a^{\alpha\beta}(y) = \mathbf{a}^\alpha(y) \cdot \mathbf{a}^\beta(y)$, or equivalently, by $(a^{\alpha\beta}(y)) = (a_{\sigma\tau}(y))^{-1}$, for all $y \in \bar{\omega}$. The covariant components of the second fundamental form of S are defined by $b_{\alpha\beta}(y) = -\partial_\alpha \mathbf{a}_3(y) \cdot \mathbf{a}_\beta(y) = \partial_\alpha \mathbf{a}_\beta(y) \cdot \mathbf{a}_3(y)$ for all $y \in \bar{\omega}$, and the mixed components of the same form are defined by $b_\alpha^\tau = -\partial_\alpha \mathbf{a}_3(y) \cdot \mathbf{a}^\tau(y) = \partial_\alpha \mathbf{a}^\tau(y) \cdot \mathbf{a}_3(y)$ for all $y \in \bar{\omega}$, or equivalently, by $b_\alpha^\tau = a^{\tau\beta} b_{\alpha\beta}$ for all $y \in \bar{\omega}$. The Christoffel symbols are defined by

$$\Gamma_{\alpha\beta}^\tau := \frac{1}{2} a^{\tau\nu}(\partial_\alpha a_{\beta\nu} + \partial_\beta a_{\alpha\nu} - \partial_\nu a_{\alpha\beta}) = \Gamma_{\beta\alpha}^\tau \quad \text{in } \bar{\omega}.$$

The two fundamental forms satisfy the *Gauss and Codazzi–Mainardi equations*:

$$R_{\cdot\alpha\sigma\tau}^\nu = b_{\tau\alpha} b_\sigma^\nu - b_{\sigma\alpha} b_\tau^\nu \quad \text{and} \quad \partial_\sigma b_{\tau\alpha} - \partial_\tau b_{\sigma\alpha} + \Gamma_{\tau\alpha}^\mu b_{\sigma\mu} - \Gamma_{\sigma\alpha}^\mu b_{\tau\mu} = 0,$$

where

$$R_{\cdot\alpha\sigma\tau}^\nu := \partial_\sigma \Gamma_{\tau\alpha}^\nu - \partial_\tau \Gamma_{\sigma\alpha}^\nu + \Gamma_{\tau\alpha}^\mu \Gamma_{\sigma\mu}^\nu - \Gamma_{\sigma\alpha}^\mu \Gamma_{\tau\mu}^\nu$$

are the mixed components of the Riemann curvature tensor associated with the metric $(a_{\alpha\beta})$.

The covariant derivatives $\eta_{\alpha|\beta} \in L^2(\omega)$ of a 1-covariant tensor field with components $\eta_\alpha \in H^1(\omega)$ are defined by $\eta_{\alpha|\beta} := \partial_\beta \eta_\alpha - \Gamma_{\beta\alpha}^\nu \eta_\nu$.

The covariant derivatives $T_{\alpha\beta|\sigma} \in H^{-1}(\omega)$ of a 2-covariant tensor field with components $T_{\alpha\beta} \in L^2(\omega)$ are defined by $T_{\alpha\beta|\sigma} := \partial_\sigma T_{\alpha\beta} - \Gamma_{\sigma\alpha}^\nu T_{\nu\beta} - \Gamma_{\sigma\beta}^\nu T_{\alpha\nu}$.

The covariant derivatives $T_{\alpha\beta\sigma|\tau} \in H^{-2}(\omega)$ of a 3-covariant tensor field with components $T_{\alpha\beta\sigma} \in H^{-1}(\omega)$ are defined by

$$T_{\alpha\beta\sigma|\tau} := \partial_\tau T_{\alpha\beta\sigma} - \Gamma_{\tau\alpha}^\nu T_{\nu\beta\sigma} - \Gamma_{\tau\beta}^\nu T_{\alpha\nu\sigma} - \Gamma_{\tau\sigma}^\nu T_{\alpha\beta\nu}.$$

The Codazzi–Mainardi equations are equivalently expressed in terms of the covariant derivatives as $b_{\alpha\sigma|\tau} = b_{\alpha\tau|\sigma}$, or equivalently, as $b_\sigma^\alpha|_\tau = b_\tau^\alpha|_\sigma$, where the covariant derivatives $b_\sigma^\alpha|_\tau$ are defined by $b_\sigma^\alpha|_\tau := \partial_\tau b_\sigma^\alpha - \Gamma_{\tau\sigma}^\mu b_\mu^\sigma + \Gamma_{\tau\mu}^\alpha b_\sigma^\mu$.

The second-order covariant derivatives $\eta_{\alpha|\sigma\tau} \in H^{-1}(\omega)$ of a 1-covariant tensor field with components $\eta_\alpha \in H^1(\omega)$ are defined by $\eta_{\alpha|\sigma\tau} := \partial_\tau \eta_{\alpha|\sigma} - \Gamma_{\tau\alpha}^\nu \eta_{\nu|\sigma} - \Gamma_{\tau\sigma}^\nu \eta_{\alpha|\nu}$, and they satisfy the *Ricci identities*

$$\eta_{\alpha|\sigma\tau} - \eta_{\alpha|\tau\sigma} = R_{\cdot\alpha\sigma\tau}^\nu \eta_\nu.$$

The second-order covariant derivatives $T_{\alpha\beta|\sigma\tau} \in H^{-2}(\omega)$ of a 2-covariant tensor field with components $T_{\alpha\beta} \in L^2(\omega)$ are defined by

$$T_{\alpha\beta|\sigma\tau} := \partial_\tau T_{\alpha\beta|\sigma} - \Gamma_{\tau\alpha}^\nu T_{\nu\beta|\sigma} - \Gamma_{\tau\beta}^\nu T_{\alpha\nu|\sigma} - \Gamma_{\tau\sigma}^\nu T_{\alpha\beta|\nu},$$

and they satisfy the *Ricci identities*

$$T_{\alpha\beta|\sigma\tau} - T_{\alpha\beta|\tau\sigma} = R_{\cdot\alpha\sigma\tau}^\mu T_{\mu\beta} + R_{\cdot\beta\sigma\tau}^\mu T_{\alpha\mu}.$$

Detailed proofs of the results announced in this Note are given in [3].

2. Saint Venant equations on a surface

Let ω be a bounded open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{R}^2)$ be an immersion. The vector fields $\mathbf{a}_i \in \mathcal{C}^2(\bar{\omega}; \mathbb{R}^3)$ and $\mathbf{a}^i \in \mathcal{C}^2(\bar{\omega}; \mathbb{R}^3)$ are defined as in Section 1. With every vector field $\boldsymbol{\eta} \in H^1(\omega; \mathbb{R}^3)$, we associate the *linearized change of metric* tensor field $(\gamma_{\alpha\beta}(\boldsymbol{\eta}))$, defined by

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\alpha \boldsymbol{\eta} \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \partial_\beta \boldsymbol{\eta}) = \gamma_{\beta\alpha}(\boldsymbol{\eta}),$$

and the *linearized change of curvature* tensor field $(\rho_{\alpha\beta}(\boldsymbol{\eta}))$, defined by

$$\rho_{\alpha\beta}(\boldsymbol{\eta}) := (\partial_{\alpha\beta} \boldsymbol{\eta} - \Gamma_{\alpha\beta}^\nu \partial_\nu \boldsymbol{\eta}) \cdot \mathbf{a}_3 = \rho_{\beta\alpha}(\boldsymbol{\eta}).$$

Note that $\gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ and $\rho_{\alpha\beta}(\boldsymbol{\eta}) \in H^{-1}(\omega)$.

The next theorem shows that these tensors necessarily satisfy specific compatibility relations, which constitute the *Saint Venant equations on a surface*. The proof rests on careful computations (all of which need to be justified in the sense of distributions), which notably use the Ricci identities and the Gauss equations.

Theorem 2.1. *The linearized change of metric tensor* $(\gamma_{\alpha\beta}) := (\gamma_{\alpha\beta}(\boldsymbol{\eta})) \in L^2(\omega; \mathbb{S}^2)$ *and the linearized change of curvature tensor* $(\rho_{\alpha\beta}) := (\rho_{\alpha\beta}(\boldsymbol{\eta})) \in H^{-1}(\omega; \mathbb{S}^2)$ *associated with a vector field* $\boldsymbol{\eta} \in H^1(\omega; \mathbb{R}^3)$ *necessarily satisfy*

$$\gamma_{\sigma\alpha|\beta\tau} + \gamma_{\tau\beta|\alpha\sigma} - \gamma_{\tau\alpha|\beta\sigma} - \gamma_{\sigma\beta|\alpha\tau} + R^{\nu}_{\cdot\alpha\sigma\tau}\gamma_{\beta\nu} - R^{\nu}_{\cdot\beta\sigma\tau}\gamma_{\alpha\nu} = b_{\tau\alpha}\rho_{\sigma\beta} + b_{\sigma\beta}\rho_{\tau\alpha} - b_{\sigma\alpha}\rho_{\tau\beta} - b_{\tau\beta}\rho_{\sigma\alpha},$$
$$\rho_{\sigma\alpha|\tau} - \rho_{\tau\alpha|\sigma} = b^{\nu}_{\sigma}(\gamma_{\alpha\nu|\tau} + \gamma_{\tau\nu|\alpha} - \gamma_{\tau\alpha|\nu}) - b^{\nu}_{\tau}(\gamma_{\alpha\nu|\sigma} + \gamma_{\sigma\nu|\alpha} - \gamma_{\sigma\alpha|\nu})$$

in the distributional sense.

3. Recovery of a vector field from its linearized change of metric and change of curvature tensors

We now characterize those symmetric matrix fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ that together satisfy the Saint Venant equations on a surface:

Theorem 3.1. *Let* ω *be a simply-connected domain in* $\mathbb{R}^2$ *and let* $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{R}^3)$ *be an immersion. Let there be given two symmetric matrix fields* $(\gamma_{\alpha\beta})$ *and* $(\rho_{\alpha\beta})$ *in the space* $L^2(\omega; \mathbb{S}^2)$ *that together satisfy in the distributional sense the Saint Venant equations on a surface* (Theorem 2.1).

Then there exists a vector field $\boldsymbol{\eta} \in H^1(\omega; \mathbb{R}^3)$ *such that*

$$\gamma_{\alpha\beta} = \frac{1}{2}(\partial_\alpha \boldsymbol{\eta} \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \partial_\beta \boldsymbol{\eta}) \quad \text{in } L^2(\omega),$$
$$\rho_{\alpha\beta} = (\partial_{\alpha\beta}\boldsymbol{\eta} - \Gamma^{\nu}_{\alpha\beta}\partial_\nu \boldsymbol{\eta}) \cdot \mathbf{a}_3 \quad \text{in } H^{-1}(\omega).$$

Sketch of proof. The proof comprises three steps. One first shows that the Saint Venant equations on a surface imply that the system

$$\lambda_{\alpha\beta|\sigma} + b_{\alpha\sigma}\lambda_\beta - b_{\beta\sigma}\lambda_\alpha = \gamma_{\sigma\beta|\alpha} - \gamma_{\sigma\alpha|\beta},$$
$$\lambda_{\alpha|\sigma} + b^{\nu}_{\sigma}\lambda_{\alpha\nu} = \rho_{\sigma\alpha} - b^{\nu}_{\sigma}\gamma_{\alpha\nu}$$

has a solution $(\lambda_{\alpha\beta}) \in L^2(\omega; \mathbb{A}^2)$ and $(\lambda_\alpha) \in L^2(\omega; \mathbb{R}^2)$. To this end, one uses a series of careful computations, coupled with an application of Poincaré's theorem in its weak form (Theorem 1.1).

Second, one shows that the symmetry of the matrix fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ imply that there exists a solution $\boldsymbol{\eta} \in H^1(\omega; \mathbb{R}^3)$ to the system

$$\partial_\alpha \boldsymbol{\eta} = (\gamma_{\alpha\beta} + \lambda_{\alpha\beta})\mathbf{a}^\beta + \lambda_\alpha \mathbf{a}^3.$$

The existence of such a vector field $\boldsymbol{\eta}$ is again obtained by an application of Poincaré's theorem in its weak form.

Finally, one shows that the symmetry of the matrix fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$, together with the antisymmetry of the matrix fields $(\lambda_{\alpha\beta})$, imply that the vector field $\boldsymbol{\eta}$ is indeed related to the tensors $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ as indicated in the statement of the theorem. This part involves straightforward computations, which notably use the well-known relations $\partial_\alpha \mathbf{a}^\sigma = -\Gamma^{\sigma}_{\alpha\mu}\mathbf{a}^\mu + b^{\sigma}_{\alpha}\mathbf{a}^3$. □

Note that the uniqueness result established in Ciarlet and C. Mardare [4, Theorem 3] shows that any vector field $\tilde{\boldsymbol{\eta}} \in H^1(\omega; \mathbb{R}^3)$ that satisfies

$$\gamma_{\alpha\beta} = \frac{1}{2}(\partial_\alpha \tilde{\boldsymbol{\eta}} \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \partial_\beta \tilde{\boldsymbol{\eta}}) \quad \text{in } L^2(\omega),$$
$$\rho_{\alpha\beta} = (\partial_{\alpha\beta}\tilde{\boldsymbol{\eta}} - \Gamma^{\nu}_{\alpha\beta}\partial_\nu \tilde{\boldsymbol{\eta}}) \cdot \mathbf{a}_3 \quad \text{in } H^{-1}(\omega),$$

is necessarily of the form

$$\tilde{\boldsymbol{\eta}}(y) = \boldsymbol{\eta}(y) + \big(\mathbf{a} + \mathbf{b} \wedge \boldsymbol{\theta}(y)\big) \quad \text{for almost all } y \in \omega,$$

where $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^3$.

4. The linearized Gauss and Codazzi–Mainardi equations

We now show that the Saint Venant equations on a surface are nothing but an infinitesimal version of the Gauss and Codazzi–Mainardi equations. These equations are recalled in the next theorem, which is a straightforward extension of a well-known result for smoother immersions $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{R}^3)$.

Theorem 4.1. *Let ω be a domain in $\mathbb{R}^2$, let $\boldsymbol{\theta} \in W^{2,p}_{\mathrm{loc}}(\omega; \mathbb{R}^3)$ be an immersion, and let the matrix fields $(a_{\alpha\beta}) \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in L^p_{\mathrm{loc}}(\omega; \mathbb{S}^2)$, $p > 2$, be defined by*

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \quad \textit{and} \quad b_{\alpha\beta} = \partial_\alpha \mathbf{a}_\beta \cdot \mathbf{a}_3 \quad \textit{in } \omega,$$

where

$$\mathbf{a}_\alpha := \partial_\alpha \boldsymbol{\theta} \quad \textit{and} \quad \mathbf{a}_3 := \frac{\mathbf{a}_1 \wedge \mathbf{a}_2}{|\mathbf{a}_1 \wedge \mathbf{a}_2|}.$$

Then the functions $a_{\alpha\beta}$ and $b_{\alpha\beta}$ together satisfy the Gauss and Codazzi–Mainardi equations in the distributional sense, viz.,

$$\begin{aligned}
&R^{\nu}_{\cdot\alpha\sigma\tau} := \partial_\sigma \Gamma^\nu_{\alpha\tau} - \partial_\tau \Gamma^\nu_{\alpha\sigma} + \Gamma^\varphi_{\alpha\tau}\Gamma^\nu_{\varphi\sigma} - \Gamma^\varphi_{\alpha\sigma}\Gamma^\nu_{\varphi\tau} = b_{\alpha\tau} b^\nu_\sigma - b_{\alpha\sigma} b^\nu_\tau \quad \textit{in } \mathcal{D}'(\omega),\\
&\partial_\sigma b_{\alpha\tau} - \partial_\tau b_{\alpha\sigma} + \Gamma^\mu_{\alpha\tau} b_{\mu\sigma} - \Gamma^\mu_{\alpha\sigma} b_{\mu\tau} = 0 \quad \textit{in } \mathcal{D}'(\omega).
\end{aligned} \tag{1}$$

As shown in Theorem 9 in S. Mardare [5], the converse of Theorem 4.1 is also true:

Theorem 4.2. *Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$ and let $a_{\alpha\beta} \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{S}^2_>)$ and $b_{\alpha\beta} \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{S}^2)$, $p > 2$, be two matrix fields that satisfy the Gauss and Codazzi–Mainardi equations* (Theorem 4.1) *in the distributional sense.*

Then there exists an immersion $\boldsymbol{\theta} \in W^{2,p}_{\mathrm{loc}}(\omega; \mathbb{R}^3)$ such that

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \quad \textit{and} \quad b_{\alpha\beta} = \partial_\alpha \mathbf{a}_\beta \cdot \mathbf{a}_3 \quad \textit{in } \omega,$$

where

$$\mathbf{a}_\alpha := \partial_\alpha \boldsymbol{\theta} \quad \textit{and} \quad \mathbf{a}_3 := \frac{\mathbf{a}_1 \wedge \mathbf{a}_2}{|\mathbf{a}_1 \wedge \mathbf{a}_2|}.$$

Our last objective is to show that Theorems 2.1 and 3.1 are in fact 'infinitesimal' versions of Theorems 4.1 and 4.2, respectively. To this end, we will show that the Saint Venant equations on a surface coincide with the *linearized* Gauss and Codazzi–Mainardi equations:

Theorem 4.3. *Let ω be an open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{R}^3)$ be an immersion. For some $p > 2$, let there be given symmetric matrix fields $(\gamma_{\alpha\beta}) \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{S}^2)$ and $(\rho_{\alpha\beta}) \in L^p_{\mathrm{loc}}(\omega; \mathbb{S}^2)$ such that the matrix fields $(a_{\alpha\beta} + \varepsilon\gamma_{\alpha\beta})$ and $(b_{\alpha\beta} + \varepsilon\rho_{\alpha\beta})$ satisfy the Gauss and Codazzi–Mainardi equations for $|\varepsilon| > 0$ small enough.*

Then the linear part with respect to ε in the Gauss and Codazzi–Mainardi equations associated with the matrix fields $(a_{\alpha\beta} + \varepsilon\gamma_{\alpha\beta})$ and $(b_{\alpha\beta} + \varepsilon\rho_{\alpha\beta})$ coincide with the Saint Venant equations on the surface $S = \boldsymbol{\theta}(\bar{\omega})$ (Theorem 2.1).

Sketch of proof. It suffices to prove that the linearized Gauss and Codazzi–Mainardi equations coincide with the Saint Venant equations on every compact subset of ω. Hence we may assume that $(\gamma_{\alpha\beta}) \in W^{1,p}(\omega; \mathbb{S}^2)$ and $(\rho_{\alpha\beta}) \in L^p(\omega; \mathbb{S}^2)$. For all ε, define the matrix fields

$$\begin{aligned}
&\big(a_{\alpha\beta}(\varepsilon)\big) := (a_{\alpha\beta}) + \varepsilon(\gamma_{\alpha\beta}) \in W^{1,p}\big(\omega; \mathbb{S}^2\big),\\
&\big(b_{\alpha\beta}(\varepsilon)\big) := (b_{\alpha\beta}) + \varepsilon(\rho_{\alpha\beta}) \in L^p\big(\omega; \mathbb{S}^2\big).
\end{aligned}$$

Since $W^{1,p}(\omega) \subset \mathcal{C}^0(\bar{\omega})$ by the Sobolev embedding theorem, there exists a number $\varepsilon_0 > 0$ such that, for all $0 \leqslant |\varepsilon| < \varepsilon_0$, the matrix field $(a_{\alpha\beta}(\varepsilon))$ is positive definite in $\bar{\omega}$. This implies that $a^{\sigma\tau}(\varepsilon) \in W^{1,p}(\omega)$, where $(a^{\sigma\tau}(\varepsilon)) = (a_{\alpha\beta}(\varepsilon))^{-1}$. Hence the Christoffel symbols

$$\Gamma_{\alpha\beta\sigma}(\varepsilon) := \frac{1}{2}\big\{\partial_\alpha a_{\sigma\beta}(\varepsilon) + \partial_\beta a_{\alpha\sigma}(\varepsilon) - \partial_\sigma a_{\alpha\beta}(\varepsilon)\big\} \quad \text{and} \quad \Gamma^\tau_{\alpha\beta}(\varepsilon) := a^{\tau\sigma}(\varepsilon)\Gamma_{\alpha\beta\sigma}(\varepsilon)$$

and the mixed components $b_\alpha^\tau(\varepsilon) = a^{\tau\beta} b_{\alpha\beta}(\varepsilon)$ of the second fundamental form all belong to the space $L^p(\omega)$. This property implies that the Gauss and Codazzi–Mainardi equations associated with the two fields $(a_{\alpha\beta}(\varepsilon))$ and $(b_{\alpha\beta}(\varepsilon))$ are well defined in the space of distributions.

With self-explanatory notations, the Gauss and Codazzi–Mainardi equations assert that, for all $|\varepsilon|$ small enough,

$$R^{\beta}_{\cdot\alpha\sigma\tau}(\varepsilon) = b_{\alpha\tau}(\varepsilon) b^\beta_\sigma(\varepsilon) - b_{\alpha\sigma}(\varepsilon) b^\beta_\tau(\varepsilon),$$
$$\partial_\sigma b_{\alpha\tau}(\varepsilon) - \partial_\tau b_{\alpha\sigma}(\varepsilon) + \varGamma^\mu_{\alpha\tau}(\varepsilon) b_{\mu\sigma}(\varepsilon) - \varGamma^\mu_{\alpha\sigma}(\varepsilon) b_{\mu\tau}(\varepsilon) = 0.$$

In order to compute the linear part of the Gauss and Codazzi–Mainardi equations associated with the fields $(a_{\alpha\beta}(\varepsilon))$ and $(b_{\alpha\beta}(\varepsilon))$, we thus proceed by expanding all the above functions as power series in ε, taking into account that the fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ also satisfy the Gauss and Codazzi–Mainardi equations in ω. This part of the proof essentially rests on a fairly lengthy series of careful computations, involving in particular the Ricci identities. $\square$

Note that, in Theorem 4.3, the field $(\gamma_{\alpha\beta})$ belongs to the space $W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{S}^2)$, so as to guarantee that $(a_{\alpha\beta}(\varepsilon)) \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{S}^2)$, which is the minimal regularity assumption under which the Riemannian curvature tensor $R_{\beta\alpha\sigma\tau}(\varepsilon)$ is well defined in the space of distributions. By contrast, the Saint Venant equations can be extended by continuity to matrix fields $(\gamma_{\alpha\beta})$ that belong only to the space $L^2_{\mathrm{loc}}(\omega; \mathbb{S}^2)$.

In [3], we also show how the results of this Note can be used to explicitly describe, and mathematically justify, an *intrinsic theory of linearly elastic shells*; cf. [2].

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References

[1] P.G. Ciarlet, P. Ciarlet Jr., Another approach to linearized elasticity and a new proof of Korn's inequality, Math. Models Methods Appl. Sci. 15 (2005) 259–271.
[2] P.G. Ciarlet, L. Gratie, A new approach to linear shell theory, Math. Models Methods Appl. Sci. 15 (2005) 1181–1202.
[3] P.G. Ciarlet, L. Gratie, C. Mardare, M. Shen, Saint Venant compatibility equations on a surface – application to intrinsic shell theory, preprint.
[4] P.G. Ciarlet, C. Mardare, On rigid and infinitesimal rigid displacements in shell theory, J. Math. Pures Appl. 83 (2004) 1–15.
[5] S. Mardare, On Pfaff systems with L^p coefficients and their applications in differential geometry, J. Math. Pures Appl. 84 (2005) 1659–1692.