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Algebraic Geometry

Invariant count of holomorphic disks in the cotangent bundles of the two-sphere and real projective plane

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Abstract

We first define enumerative invariants of the cotangent bundles of the two-sphere and real projective plane. These invariants are obtained in the framework of symplectic field theory by counting with respect to some sign holomorphic disks with punctures sitting on the zero section. Then, we relate these invariants with the ones of closed real symplectic four-manifolds which have been constructed earlier. This relation provides some congruences and recursive formulas for the latter as well as sharpness results for the associated lower bounds in real enumerative geometry. ***To cite this article: J.-Y. Welschinger, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Comptage invariant de disques holomorphes dans les fibrés cotangents de la sphère de dimension deux et du plan projectif réel. Dans une première partie, nous introduisons des invariants énumératifs des fibrés cotangents de la sphère de dimension deux et du plan projectif réel. Ces invariants sont obtenus dans le langage de la théorie symplectique des champs en comptant en fonction d'un signe les disques holomorphes avec pointes qui reposent sur la section nulle. Puis nous relions ces invariants avec ceux des variétés symplectiques réelles de dimension quatre précédemment construits et déduisons des congruences et formules récurrentes pour ces derniers, ainsi que des résultats d'optimalité pour les bornes inférieures associées en géométrie énumérative réelle. ***Pour citer cet article : J.-Y. Welschinger, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

1. Enumerative invariants of cotangent bundles

Let L be a two-dimensional sphere or a real projective plane. Let λ be the Liouville form on T^*L and c_L be the involution $(q, p) \in T^*L \mapsto (q, -p) \in T^*L$. The latter satisfies $c_L^*\lambda = -\lambda$ so that $d\lambda$ is a symplectic form on T^*L for which c_L is anti-symplectic. Let g be a Riemannian metric with constant curvature on L, U^*L be the set of $(q, p) \in T^*L$ for which $g(p, p) \leqslant 1$ and S^*L be the boundary of U^*L. The restriction of λ to S^*L is a contact form and we denote by R_λ the associated Reeb vector field. The flow generated by R_λ is nothing but the geodesic

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flow so that all orbits are periodic with the same minimal period. Following the framework of symplectic field theory, we denote by $\mathcal{J}_\lambda$ the space of almost complex structures which are tamed by $d\lambda$ and asymptotically cylindrical. More precisely, let us fix once and for all an identification of the complement of the zero section L of T^*L with the symplectisation $(\mathbb{R} \times S^*L, d(\mathrm{e}^\rho \lambda))$ of (S^*L, λ). We then denote by $\mathcal{J}_\lambda$ the space of almost complex structures J of class C^l of T^*L, $l \gg 1$, which are tamed by $d\lambda$, satisfy $J(\frac{\partial}{\partial\rho}) = R_\lambda$ for $\rho \gg 1$, where ρ denotes the $\mathbb{R}$-coordinate in the symplectisation $\mathbb{R} \times S^*L$, and are invariant under translation by ρ after some given rank ρ_0. We denote by $\mathbb{R}\mathcal{J}_\lambda \subset \mathcal{J}_\lambda$ the subspace of almost complex structures for which c_L is J-antiholomorphic. Note that $\mathcal{J}_\lambda$ and $\mathbb{R}\mathcal{J}_\lambda$ are both separable Banach manifolds which are non-empty and contractible.

Now, for a generic $J \in \mathbb{R}\mathcal{J}_\lambda$ we are going to count J-holomorphic disks with some given number of punctures that are properly immersed in T^*L such that the boundaries of the disks are mapped to the zero section. Remember that from Theorem 1.2 of [4] and [1], such a disk converges asymptotically near each of its punctures to some orbit of the Reeb flow travelled around some number of times. We call this number of times the multiplicity of the asymptotic orbit. In order to get only finitely many such disks, these disks will be subject to some constraints. Either some of the asymptotic orbits will be prescribed, or these disks will have to pass through some points of L or of $T^*L \setminus L$. Let e_i, $i \geqslant 1$, be the sequence of integers which do not vanish only at the ith rank where it equals one. Let $\alpha = \sum_{i \in \mathbb{N}^*} \alpha_i e_i$ and $\beta = \sum_{i \in \mathbb{N}^*} \beta_i e_i$ be two sequences of non-negative integers which vanish after some given rank. These two sequences encode the number of prescribed and non-prescribed asymptotic orbits respectively together with their multiplicities $i \in \mathbb{N}^*$. The number of punctures is thus $\delta = \sum_{i \in \mathbb{N}^*} (\alpha_i + \beta_i)$ and we choose a set Γ of $\sum_{i \in \mathbb{N}^*} \alpha_i$ disjoint closed geodesics of L to prescribe the asymptotic orbits. Now to fix the point constraints, let $r \in \mathbb{N}$ and $x_1, \ldots, x_r$ be r distinct points of L. Likewise, let $r_L \in \mathbb{N}$ and $\xi_1, \bar{\xi}_1, \ldots, \xi_{r_L}, \bar{\xi}_{r_L}$ be r_L pairs of distinct points of $T^*L \setminus L$ such that $c_L(\xi_i) = \bar{\xi}_i$. We assume that

$$r + 2r_L + 2\#\Gamma = 2\delta + \epsilon \sum_{i \in \mathbb{N}^*} i(\alpha_i + \beta_i) - 1, \tag{1}$$

where $\epsilon = 2$ if L is homeomorphic to a sphere, and $\epsilon = 1$ if L is homeomorphic to a real projective plane. Then, for any generic almost complex structure $J \in \mathbb{R}\mathcal{J}_\lambda$, there are only finitely many disks with δ punctures that are properly mapped to T^*L in such a way that the boundary of the disk is mapped to the zero section L, it passes through $\underline{x}$ and intersects each pair $\{\xi_i, \bar{\xi}_i\}$, and for $j \in \mathbb{N}^*$, the disk is asymptotic to α_j periodic orbits of the Reeb flow lifting some geodesic of the set Γ and β_j other non-prescribed periodic orbits, each of these orbits being of multiplicity j. Denote by $\mathcal{D}(\alpha, \beta, \Gamma, \underline{x}, \underline{\xi}, J)$ this finite set of disks. It follows from the generic choice of J that all these disks are immersed. For such a disk $D \in \mathcal{D}(\alpha, \beta, \Gamma, \underline{x}, \underline{\xi}, J)$, denote by $m(D)$ the finite number of transversal intersection points of the interior of the disk D with the zero section L and call this number the mass of the disk. Next, set

$$F_{(r,r_L)}(\alpha, \beta, \Gamma, \underline{x}, \underline{\xi}, J) = \frac{1}{2} \sum_{D \in \mathcal{D}(\alpha, \beta, \Gamma, \underline{x}, \underline{\xi}, J)} (-1)^{m(D)} \in \mathbb{Z}.$$

Note that the disks of $\mathcal{D}(\alpha, \beta, \Gamma, \underline{x}, \underline{\xi}, J)$ come in pairs exchanged by the involution c_L, that is why the coefficient $\frac{1}{2}$ does not prevent $F_{(r,r_L)}(\alpha, \beta, \Gamma, \underline{x}, \underline{\xi}, J)$ from being an integer.

Theorem 1.1. *Let α, β be two sequences of non-negative integers which vanish after some given rank. Choose as above a set Γ of prescribed asymptotic orbits and sets $\underline{x}$ and $\underline{\xi}$ of r and r_L point constraints in L and $T^*L \setminus L$ respectively, where this number of constraints satisfies* (1). *Then, the number $F_{(r,r_L)}(\alpha, \beta, \Gamma, \underline{x}, \underline{\xi}, J)$ of holomorphic disks with $\delta = \sum_{i \in \mathbb{N}^*} (\alpha_i + \beta_i)$ punctures satisfying these constraints and counted with respect to their mass neither depends on the choice of constraints $\Gamma, \underline{x}, \underline{\xi}$, nor on the generic choice of the almost complex structure $J \in \mathbb{R}\mathcal{J}_\lambda$.*

The proof of Theorem 1.1 goes basically along the same lines as the proof of Theorem 2.1 of [10]. This Theorem 1.1 provides enumerative invariants of the cotangent bundles of the two-sphere and the real projective plane which will be denoted by $F_{(r,r_L)}(\alpha, \beta)$, and when r_L vanishes by $F(\alpha, \beta)$. By using similar methods as the ones used in §3.4 of [10], one can obtain the following values:

Lemma 1.2. *If L is homeomorphic to a sphere and $r_L = 0$, then $F(e_1, 0) = F(0, e_1) = 1$, $F(e_2, 0) = 2$, $F(0, e_2) = 8$, $F(2e_1, 0) = 2$, $F(e_1, e_1) = 4$ and $F(0, 2e_1) = 6$.*

Lemma 1.3. *If L is homeomorphic to a real projective plane and $r_L = 0$, then $F(e_1, 0) = F(0, e_1) = F(e_2, 0) = F(2e_1, 0) = F(e_1, e_1) = F(0, 2e_1) = 1$ and $F(0, e_2) = 4$.*

Lemma 1.4. *If L is homeomorphic to a real projective plane and $r_L = 0$, then $F(e_3, 0) = 2$, $F(0, e_3) = 12$, $F(e_1 + e_2, 0) = 2$, $F(e_1, e_2) = 8$, $F(e_2, e_1) = 4$, $F(0, e_1 + e_2) = 24$, $F(3e_1, 0) = 2$, $F(2e_1, e_1) = 4$, $F(e_1, 2e_1) = 6$ and $F(0, 3e_1) = 8$.*

Remark 1. Further constructions of such invariants are in process.

2. Relation with invariants of closed real symplectic four-manifolds

Let (X, ω, c_X) be a closed real symplectic four-manifold. In [10], we have constructed enumerative invariants which when the real locus $\mathbb{R}X = \mathrm{fix}(c_X)$ is connected, take the form of a function $\chi : d \in H_2(X; \mathbb{Z}) \mapsto \chi^d(T) = \sum_{r=0}^{c_1(X)d-1} \chi_r^d(T) T^r \in \mathbb{Z}[T]$. When the real locus $\mathbb{R}X$ contains a sphere or a real projective plane, it is possible to relate these invariants with the invariants $F(\alpha, \beta)$ defined in Theorem 1.1 by using the splitting principle of symplectic field theory, see [2]. We present here a few applications of this point of view and then briefly discuss how to obtain them.

Proposition 2.1. *If (X, ω, c_X) is the ellipsoid quadric, $d \in H_2(X; \mathbb{Z})$ is such that $c_1(X)d \geqslant 1$, and $r, r_X \in \mathbb{N}$ such that $r + 2r_X = c_1(X)d - 1$, then, two to the power $max(0, r_X - \frac{1}{2}(r + 1))$ divides χ_r^d. If (X, ω, c_X) is the complex projective plane, $d > 0$ and $r, r_X \in \mathbb{N}$ is such that $r + 2r_X = 3d - 1$, then, two to the power* $\max(0, r_X - r - 1)$ *divides χ_r^d.*

Proposition 2.2. *Let (X, ω, c_X) be the projective plane or the ellipsoid quadric, $d \in H_2(X; \mathbb{Z})$ be a positive multiple of the hyperplane section for the standard embedding and $0 \leqslant r \leqslant 1$. Then $(-1)^{\frac{1}{2}(d^2 - c_1(X)d + 2)} \chi_r^d \geqslant 0$. Moreover, the lower bounds given by Corollary 2.2 of* [10] *are sharp in these cases, namely realized by any generic almost complex structure $J \in \mathbb{R}\mathcal{J}_\lambda$ having a sufficiently long neck near the real part. Finally, the invariant vanishes only when the positive multiple of the hyperplane section is two in the case of the ellipsoid and three and four in the case of the projective plane.*

Remark 2. From Proposition 2.2 we get in particular that in the case of the complex projective plane, the invariants χ_0^d for $d = 3 \bmod (4)$, $d \neq 3$, and χ_1^d for $d = 0 \bmod (4)$, $d \neq 4$, are negative. This disproves Conjecture 6 of [5] on non-negativity of these invariants.

In order to prove Propositions 2.1 and 2.2, we stretch the neck of a generic almost complex structure of (X, ω, c_X) near the real part until the manifold splits in two pieces T^*L and $X \setminus L$, where $L = \mathbb{R}X$. The real rational curves which are counted by χ_r^d then split into rational curves with punctures and finite Hofer energy in these two pieces. The main parts of these curves in T^*L consist of real rational curves which pass through the points $x_1, \ldots, x_r$, these parts are precisely counted by the invariant $F(\alpha, \beta)$, where α, β can take finitely many values but r is fixed, given by χ_r^d. The parts in $X \setminus L$ are relative invariants arising in symplectic field theory which have nothing to do with the real structure c_L. However, these parts come in pairs exchanged by c_L and have to pass through pairs of complex conjugated points. The power of two arising in Proposition 2.1 basically comes from the number of ways to divide these pairs into two disjoint sets exchanged by c_L. Sharpness of the lower bounds given by Proposition 2.2 comes from the fact that in these cases, all the split curves have the same sign which means that for an almost complex structure with a very long neck, close to the splitting, all the real rational curves counted by χ_r^d have masses of the same parity. Hence, this number of curves precisely equals χ_r^d. Note that in the case of the complex projective plane or the ellipsoid, $X \setminus L$ is just a complex line bundle of degree four or two respectively over $\mathbb{C}P^1$ – for the standard complex structure. The rational holomorphic curves of finite Hofer energy of $X \setminus L$ thus compactify into irreducible curves of the rational ruled surface of degree four or two respectively, having tangency conditions with the exceptional divisor predicted by the sequences α, β. Recursive formulas for such relative invariants have been obtained, see Theorem 6.8 of [9]. In particular for these surfaces, as soon as the invariants $F(\alpha, \beta)$ are computed for a given r, this process provides a recursive formula to compute all the invariants χ_r^d for this r. From Lemmas 1.2, 1.3 and 1.4, we thus deduce recursive formulas computing the invariants χ_r^d for $r \leqslant 2$ in the projective plane and $r \leqslant 3$ in the ellipsoid. Unfortunately, space is too short here to provide the explicit formulas (they will be given in a forthcoming paper). Let us provide instead the first values computed by hand from these formulas:

Proposition 2.3.

(1) *Let (X, ω, c_X) be the ellipsoid and f_1, f_2 be the homology classes of its two rulings. Then,* $\chi^{f_1+f_2}(T) = T + T^3$, $\chi^{2(f_1+f_2)}(T) = 2T^3 + 4T^5 + 6T^7$, $\chi^{3(f_1+f_2)}(T) = 16T + 16T^3 + \mathrm{o}(T^4)$, $\chi^{4(f_1+f_2)}(T) = -128T + 384T^3 + \mathrm{o}(T^4)$ *and* $\chi^{5(f_1+f_2)}(T) = 24576T + \mathrm{o}(T^2)$.
(2) *Let (X, ω, c_X) be the projective plane, then,* $\chi^4(T) = \mathrm{o}(T^2)$, $\chi^5(T) = 64 + 64T^2 + \mathrm{o}(T^3)$, $\chi^6(T) = 1024T + 1536T^3 + \mathrm{o}(T^4)$, $\chi^7(T) = -14336 + 11776T^2 + \mathrm{o}(T^3)$ *and* $\chi^8(T) = -280576T + \mathrm{o}(T^2)$.

Remark 3. The values $\chi_1^4 = 0$, $\chi_0^5 = \chi_2^5 = 64$ have already been computed in [5] thanks to a computer and the algorithm [8]. Recall that the latter extends the one given in [7] to compute the leading coefficient of $\chi^d(T)$. Note also that I. Itenberg, V. Kharlamov and E. Shustin [6] have recently produced a real version of [3] to get a recursive formula satisfied by the invariants χ_{3d-1}^d of real toric Del Pezzo surfaces.

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