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Differential Geometry

Strictly nearly Kähler 6-manifolds are not compatible with symplectic forms

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Abstract

We show that the almost complex structure underlying a non-Kähler, nearly Kähler 6-manifold (in particular, the standard almost complex structure of S^6) cannot be compatible with any symplectic form, even locally. ***To cite this article: M. Lejmi, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

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Résumé

Les variétés strictement approximativement kählérienne de dimension 6 et les formes symplectiques. Nous démontrons que la structure presque-complexe d'une variété nearly-kählérienne non-intégrable de dimension 6-en particulier la structure presque-complexe standard sur la sphère S^6-ne peut pas être compatible avec une forme symplectque. ***Pour citer cet article : M. Lejmi, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

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Every symplectic manifold (M,ω) gives rise to an infinite dimensional, contractible Fréchet space of *ω-compatible* almost complex structures, J, introduced by the property that the bilinear form $g(\cdot,\cdot)=\omega(\cdot,J\cdot)$ is symmetric and positive-definite (i.e. defines a Riemannian metric on M); in this case the (J,g) is an almost Hermitian structure on M, which is referred to as an *almost Kähler* structure compatible with ω.

It is natural to wonder whether or not a given almost complex structure J on M is ω-compatible for some symplectic form ω? This question, which was first raised and studied by J. Armstrong in [1], can be asked both locally and globally and the corresponding answers are quite different in nature. In this Note we are interested in the local aspect of the problem, namely we consider the following:

Question 1. Is a given almost complex structure J on M locally compatible with symplectic forms? In other words, given J, can one find in a neighbourhood of each point of M a symplectic form compatible with J?

Following [13], we shall refer to almost complex structures which are locally compatible with symplectic forms as almost complex structures having the *local symplectic property*.

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As a trivial example, any *integrable* almost complex structure satisfies the local symplectic property (although there are many complex manifolds which are not symplectic). In particular, when M is 2-dimensional the answer of Question 1 is always positive. J. Armstrong [1, p. 10] claimed (without providing an argument) that the answer to Question 1 is always positive in 4 dimensions too.

Theorem 1. *Any almost complex manifold of dimension* 4 *has the local symplectic property.*

A proof of this result is given in [12, Lemma A.1]. For the sake of completeness, and since we find the arguments in [12] incomplete (see Remark 1 below), we give here an alternative argument based on the Malgrange existence theorem of local solutions of elliptic systems of PDE's.

Proof of Theorem 1. Let (M, J) an almost complex 4-manifold. The vector bundle of (real) 2-forms, $\wedge^2(M)$, decomposes with respect to J as a direct sum $\wedge^2(M) = \wedge^{J,+}(M) \oplus \wedge^{J,-}(M)$, where $\wedge^{J,+}(M)$ (resp. $\wedge^{J,-}(M)$) is the vector bundle of $\wedge J$-invariant 2-forms (resp. $\wedge J$-anti-invariant) 2-forms. (The vector bundle $\wedge^{J,-}(M)$, endowed with the complex structure $(\mathcal{J}\phi)(\cdot,\cdot) := -\phi(J\cdot,\cdot)$, is naturally isomorphic to the anti-canonical bundle $K_J^{-1} \cong \wedge^{0,2}(M)$ of (M, J); likewise, $\wedge^{J,+}(M) \otimes \mathbf{C} \cong \wedge^{1,1}(M)$.) We denote by $\Omega^{J,\pm}(M)$ etc. the spaces of smooth sections of the corresponding bundles. The above splitting of real 2-forms gives rise to a decomposition of the exterior derivative $d : \Omega^1(M) \to \Omega^2(M)$ as the sum of two differential operators $d^{\pm} : \Omega^1(M) \to \Omega^{J,\pm}(M)$. In order to prove Theorem 1, it is enough to show that for any point $p \in M$ there exists a (connected) neighborhood $U \ni p$ and a 1-form $\alpha \in \Omega^1(U)$, such that, on U,

$$d^-\alpha = 0, \qquad d\alpha \wedge d\alpha > 0, \tag{1}$$

where the sign of a 4-form is determined by the orientation induced by J. To solve (1), we first notice that the principal symbol of d^- is the linear map $\sigma(d^-)_\xi(\alpha) = \frac{1}{2}\big(\xi \wedge \alpha - J^*\xi \wedge J^*\alpha\big)$, where $\xi, \alpha \in T_p^*(M)$ and J^* acts on $T_p^*(M)$ by $(J^*\alpha)(X) = -\alpha(JX)$. Thus, in 4 dimensions, $\sigma(d^-)_\xi\colon \wedge_p^1(M) \mapsto \wedge_p^{J,-}(M)$ is *surjective* for any $\xi \in T_p^*M \setminus \{0\}$. We can then associate to d^- a second order *elliptic* linear differential operator $P : \Omega^{J,-}(M) \to \Omega^{J,-}(M)$ by putting $P := d^-\delta^h$, where h is some J-compatible almost Hermitian metric on (M, J) and $\delta^h : \Omega^2(M) \to \Omega^1(M)$ is the corresponding co-differential, the formal adjoint operator of d with respect to L_2-product defined by h. (The principal symbol of P is given by $\sigma(P)_\xi(\Phi) = -|\xi|^2\Phi$, $\forall \xi \in T_p^*(M), \Phi \in \wedge_p^{J,-}(M)$.)

In terms of P, we want to show that for any given point $p \in M$ one can a find a (connected) neighborhood $U \ni p$ and a J-anti-invariant 2-form $\Phi \in \Omega^{J,-}(U)$, such that

$$P(\Phi) = 0, \qquad d\delta^h(\Phi) \wedge d\delta^h(\Phi) > 0 \tag{2}$$

at any point of U. Since P is elliptic, it is enough to find a smooth 2-form $\Phi_0 \in \Omega^{J,-}(M)$, which verifies (2) only at p (i.e. an infinitesimal solution of (2)). Indeed, for any such Φ_0 one can consider the system $P(\Psi) + P(\Phi_0) = 0$. Using the implicit function theorem, it is shown in [7, p. 132] that for any $\varepsilon > 0$ there exist a neighborhood U_ε of p and a solution $\Psi_\varepsilon \in \Omega^{J,-}(U_\epsilon)$ with $\|\Psi_\varepsilon\|_{C^{2,\alpha}} < \varepsilon$ (where $\|\cdot\|$ stands for the Hölder norm of $C^{2,\alpha}(U)$). Then, for ε small enough, $\Phi = \Phi_0 + \Psi_\varepsilon$ and $U = U_\varepsilon$ will satisfy (2).

We thus reduced the problem to verifying that at each point $p \in M$ an infinitesimal solution always exists (for a suitable choice of h). Denote by $S^\ell(T_p^*(M)) \otimes \wedge_p^{J,-}(M)$ the space of ℓ-jets at p of elements of $\Omega^{J,-}(M)$ (where S^ℓ stands for the ℓth symmetric tensor power). By the Borel lemma, for any sequence $a_\ell \in S^\ell(T_p^*(M)) \otimes \wedge_p^{J,-}(M)$ ($\ell = 0, 1, \ldots$), there exists a $\Phi \in \Omega^{J,-}(M)$ whose ℓth jet at p is a_ℓ. Thus, it is enough to show that there exists jets $e = (a_2, a_1, a_0)$ such that $P(e) = 0$ and $d\delta^h(e) \wedge d\delta^h(e) > 0$, where the linear differential operators of order $\leqslant 2$ are identified with the induced linear maps on the space of jets of order $\leqslant 2$. In fact, we will seek for an e verifying the yet stronger condition $((d\delta^h(e))_0 = 0$, where $(\cdot)_0$ denotes the primitive part of a 2-form (i.e. the orthogonal projection to $F^\perp$). Clearly, $(d\delta^h(e))_0 = 0$ implies $P(e) = 0$ and $d\delta^h(e) = \frac{1}{2}L_h(e)F$, where L_h corresponds to the the linear differential operator $L_h(\Phi) := h(d\delta^h\Phi, F)$. It follows that $d\delta^h(e) \wedge d\delta^h(e) = \frac{1}{2}(L_h(e))^2 v_h$ which is positive as soon as $L_h(e) \neq 0$. A standard calculation shows that L_h is, in fact, of order one with principle symbol $\sigma_\xi(L_h)(\Phi) = -\Phi(\xi^\sharp, J\theta_h^\sharp) + 2\sum_{i=1}^4 \Phi(JN(\xi^\sharp, e_i), e_i)$, where $\theta^h := J\delta_h F$ is the Lee form of (h, J), $\sharp$ stands the isomorphism between $T^*(M)$ and $T(M)$ via h, $\{e_i\}$ is any h-orthonormal basis of $T_p(M)$ and $4N(\cdot,\cdot) = [J\cdot, J\cdot] - J[J\cdot,\cdot] - J[\cdot, J\cdot] - [\cdot,\cdot]$ is the Nijenhuis tensor of J. If necessary, one can make a conformal change $e^f h$ of h if necessary with $f(p)=0, df_p \neq 0$, in order to obtain $\sigma_\xi(L_h) \neq 0$. Thus, we can start with $e'=(a_1,a_0)$ such that $L_h(e') \neq 0$. The principal symbol of $(d\delta^h)$ is $\sigma_\xi(d\delta^h)(\Phi) = -\xi \wedge \iota_{\xi^\sharp}\Phi$. By polarization over ξ, it induces a linear map from $S^2(T_p^*(M)) \otimes \wedge_p^{J,-}(M)$ to the space of primitive 2-forms $(\wedge_p^2(M))_0$. The fact that this map is *surjective* tells us that there exists an $a_2 \in S^2(T_p^*(M)) \otimes \wedge_p^{J,-}(M)$ such that $e=(a_2,a_1,a_0)$ verifies $(d\delta^h(e))_0 = 0$. Since $L_h(e) = L_h(e') \neq 0$, this concludes the proof. $\square$

Remark 1. The argument given in [12] relies on a claim from [10] that for *any* non-degenerate 2-form Ω with $d\Omega \neq 0$, there exists a local system of coordinates (x,y,z,t) such that $\Omega = \mathrm{e}^x(dx \wedge dy + dz \wedge dt)$. We note that the existence of such coordinates implies that Ω is conformal to a symplectic form. There are, however, many non-degenerate 2-forms which do not verify the latter condition. Indeed, in 4 dimensions, to any non-degenerate 2-form Ω one can associate a 1-form θ, called the *Lee form*, such that $d\Omega = \theta \wedge \Omega$. Under a conformal transformation $\widetilde{\Omega} = \mathrm{e}^f \Omega$ the Lee form changes by $\tilde{\theta} = \theta + df$. It follows that Ω is (locally) conformally symplectic iff $d\theta = 0$. For example, the 2-form $\Omega = \mathrm{e}^{xz}\,\mathrm{d}x \wedge dy + dz \wedge dt$ is non-degenerate and has non-closed Lee form $\theta = \mathrm{e}^{xz}\,\mathrm{d}z$.

Remark 2. Theorem 3.1 in [14] affirms that there are almost complex structures on $\mathbf{R}^4$, which do *not* obey the local symplectic property. One can see that the statement is incorrect by constructing symplectic forms compatible with these almost complex structures. In fact, when the function $f(x)$ in this theorem depends on x_3 only, then the corresponding almost complex structure is even integrable.

The situation dramatically changes in dimension higher than 4. Indeed, it follows from [3] that the standard almost complex structure of S^6 does not satisfy the local symplectic property; A. Tomassini [14] gave other explicit examples of 6-dimensional almost complex manifolds which do not satisfy the local symplectic property. In dimension greater than 10, J. Armstrong [1] proved that there is an open set of (germs of) almost complex structures which does not satisfy the local symplectic property. Nevertheless, a criterion of deciding if a given almost complex structure has the local symplectic property is still to come.

We give below a negative answer to Question 1 for a special class of almost complex 6-manifolds of increasing current interest, the so-called *strictly nearly Kähler* 6-manifolds (see e.g. [4,5,8,9,11,15] and the references there in). After the submission of a first version of the manuscript, it was kindly pointed out to me by R. Bryant that this result also follows from the more general considerations in [2].

Theorem 2. *The underlying almost complex structure of a non-integrable, nearly Kähler* 6*-manifold is not compatible with any symplectic form.*

Recall that an almost Hermitian structure (h,J) is *nearly Kähler* if the covariant derivative (with respect to the Levi-Civita connection D^h) of the corresponding fundamental 2-form $F(\cdot,\cdot) = h(J\cdot,\cdot)$ satisfies $D^h F = \frac{1}{3} dF$ (nearly Kähler manifolds was first studied by A. Gray [5]). Equivalently, the Nijenhuis thensor N is related to dF by (see e.g. [6]):

$$h\big(JN(X,Y),Z\big) = \frac{1}{3} dF(X,Y,Z), \quad \forall X,Y,Z \in T(M). \tag{3}$$

Apart from the integrable case, examples include S^6 with its canonical almost complex structure and metric, the bi-invariant almost complex structure on $S^3 \times S^3$ with its 3-symmetric almost-Hermitian structure, the twistor spaces over Einstein self-dual 4-manifolds, endowed with the anti-tautological almost complex structure.

A key property of a *non-integrable* nearly Kähler 6-manifold is that the 3-form dF is the imaginary part of a nowhere vanishing complex $(3,0)$-form Ψ on (M,J) [11]. The identity (3) then reads as

$$N = \frac{1}{6} h^* \circ \Psi, \tag{4}$$

where the Nijenhuis tensor N is viewed as a linear map $N : \wedge^2(T^{1,0}(M)) \to T^{0,1}(M)$, the induced Hermitian metric h^* on (T^*M, J^*) provides an isomorphism $h^* : \wedge^{1,0}(M) \to T^{0,1}(M)$, and the complex volume form Ψ identifies $\wedge^2(T^{1,0}(M))$ with $\wedge^{1,0}(M)$.

Theorem 2 is then an immediate corollary of the following:

Proposition 1. *Let (M, J) be an almost complex 6-manifold. Suppose that at some point p the Nijenhuis tensor N does not vanish and can be written in the form*

$$N_p = h_p^* \circ \psi_p, \tag{5}$$

where $h_p^* : \wedge_p^{1,0}(M) \to T_p^{0,1}(M)$ defines a real, J_p^*-invariant, symmetric quasi-definite form on $T_p^*(M)$, and $\psi_p \in \wedge_p^{3,0}(M)$ is a non-zero $(3,0)$-form. Then, J cannot be compatible with any symplectic form defined in a neighborhood of p.

Proof of Proposition 1. Since h_p^* is J_p^*-invariant, symmetric and quasi-definite, there exists a basis $\{\alpha_1, \alpha_2, \alpha_3\}$ of $\wedge_p^{1,0}(M)$, with dual basis $\{Z_1, Z_2, Z_3\}$ of $T_p^{1,0}(M)$, such that $h_p^* = \sum_{i=1}^3 \lambda_i (Z_i \otimes \overline{Z}_i + \overline{Z}_i \otimes Z_i)$ with $\lambda_i \geqslant 0$, and $\psi_p = \alpha_1 \wedge \alpha_2 \wedge \alpha_3$. (Since N_p is not zero, at least one of the λ_i's is positive.) The condition (5) then reads as

$$N(Z_1, Z_2) = \lambda_3 \overline{Z}_3, \qquad N(Z_2, Z_3) = \lambda_1 \overline{Z}_1, \qquad N(Z_3, Z_1) = \lambda_2 \overline{Z}_2. \tag{6}$$

Suppose J is ω-compatible for some symplectic form about p. The corresponding almost Kähler structure (J, g, ω) then satisfies (see e.g. [6]): $(D_X^g \omega)(Y, Z) = -2g(JN(Y, Z), X)$, where D^g is the Levi-Civita connection of g. Taking a cyclic permutation over X, Y, Z and using the fact that ω is closed, one gets $\sigma_{X,Y,Z}(g(JN(Y, Z), X)) = 0$; with respect to the local basis verifying (6), this implies $\sqrt{-1} \sum_{i=1}^3 \lambda_i \|Z_i\|_g^2 = 0$, a contradiction. $\square$

Remark 3. The proof of Proposition 1 shows slightly more: there is no an almost Hermitian metric g, defined in a neighborhood of p, such that the fundamental 2-form ω of (g, J) satisfies $(d\omega)^{3,0} = 0$, where $(d\omega)^{3,0}$ stands for the projection of $d\omega$ to $\wedge^{3,0}(M)$.

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