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Riemann–Hilbert approach for the Camassa–Holm equation on the line

Anne Boutet de Monvel [a], Dmitry Shepelsky [b]

[a] *Institut de mathématiques de Jussieu, case 7012, université Paris 7, 2, place Jussieu, 75251 Paris cedex 05, France*
[b] *Institute for Low Temperature Physics, 47 Lenin Avenue, 61103 Kharkiv, Ukraine*

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Presented by Paul Malliavin

This Note is dedicated to Henry McKean, from whom we have learned so much.

Abstract

We present a Riemann–Hilbert problem formalism for the initial value problem for the Camassa–Holm equation $u_t - u_{txx} + 2\omega u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}$ on the line (CH). We show that: (i) for all $\omega > 0$, the solution of this problem can be obtained in a parametric form via the solution of some associated Riemann–Hilbert problem; (ii) for large time, it develops into a train of smooth solitons; (iii) for small ω, this soliton train is close to a train of peakons, which are piecewise smooth solutions of the CH equation for $\omega = 0$. ***To cite this article: A. Boutet de Monvel, D. Shepelsky, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

L'équation de Camassa–Holm sur la droite par la méthode de Riemann–Hilbert. Nous étudions par la méthode de « Riemann–Hilbert » le problème de Cauchy pour l'équation de Camassa–Holm (CH) sur la droite : $u_t - u_{txx} + 2\omega u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}$. Nous obtenons que : (i) pour tout $\omega > 0$, la solution du problème de Cauchy s'exprime de façon paramétrique en termes de la solution d'un problème de Riemann–Hilbert associé ; (ii) cette solution a pour asymptotique, pour t grand, un train de solitons lisses ; (iii) pour $\omega \to 0$, ce train de solitons tend vers un train de « peakons », solutions lisses par morceaux de l'équation CH pour $\omega = 0$. ***Pour citer cet article : A. Boutet de Monvel, D. Shepelsky, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

Nous utilisons la méthode de « scattering inverse », sous la forme d'un problème de Riemann–Hilbert, pour étudier le problème de Cauchy (1) pour l'équation de Camassa–Holm $u_t - u_{txx} + 2\omega u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}$, ω désignant un paramètre réel. La donnée initiale $u_0(x)$ est supposée C^∞ à décroissance rapide à l'infini. La formulation par un problème de Riemann–Hilbert nous permet d'obtenir le comportement asymptotique des solutions de (1) pour $t \to +\infty$ par la méthode de « plus grande pente non linéaire » développée par Deift et Zhou [8]. Dans cette Note, nous

E-mail addresses: aboutet@math.jussieu.fr (A. Boutet de Monvel), shepelsky@yahoo.com (D. Shepelsky).

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étudions l'asymptotique de la partie « solitons » pour $\omega > 0$, ainsi que sa relation, lorsque $\omega \to 0$, avec les « peakons », qui sont des solutions lisses par morceaux, de type solitons, de l'équation de Camassa–Holm de paramètre $\omega = 0$.

L'équation de Camassa–Holm (CH) est un modèle approché décrivant l'évolution d'un fluide dans un canal peu profond, sans viscosité, mais sous influence de la gravité [3]. La constante ω est liée à la vitesse critique des ondes. Il est connu [4] que, si l'on pose $m = u - u_{xx}$, l'hypothèse initiale $m(x, 0) + \omega > 0$ pour tout x implique l'existence de $m(x, t)$ pour tout temps $t > 0$, ainsi que la positivité $m(x, t) + \omega > 0$, ce qui justifie alors la forme équivalente (2) de l'équation CH.

A partir de la représentation standard (3) de la paire de Lax pour l'équation CH, nous développons un formalisme direct pour le problème de scattering, en utilisant la paire de Lax sous la forme du système linéaire du premier ordre (6), choisi de telle sorte qu'on puisse en contrôler les solutions en fonction du paramètre spectral. Nous définissons des solutions appropriées de (6) en utilisant les équations de Volterra (7) associées. Puis nous transformons la relation de scattering (8) en un problème de conjugaison analytique que l'on peut traiter comme un problème de Riemann–Hilbert dans le plan complexe du paramètre spectral. Ce problème de Riemann–Hilbert a pour données une matrice de saut le long de l'axe réel et des paramètres fixant des conditions de résidus. Pour exprimer ces données en termes des données de Cauchy, nous introduisons une échelle y (10), dans laquelle nous formulons finalement le problème de Riemann–Hilbert vectoriel associé (12).

En utilisant le fait que l'équation (3a) de la paire de Lax se trivialise pour $\lambda = 0$, nous démontrons que la solution du problème de Cauchy (1) s'exprime, sous forme paramétrique (13), en termes de la solution du problème de Riemann–Hilbert (12) associé. Puis nous étudions le comportement asymptotique et nous montrons que dans le secteur à solitons, $x > 2\omega t$, la solution du problème de Cauchy (1) évolue comme un train fini de solitons lisses. Nous montrons enfin que lorsque $\omega \to 0$, ce train de solitons tend vers le train de « peakons » associés aux valeurs propres du problème spectral (3a) relatif à la valeur $\omega = 0$.

1. Introduction

We present the inverse scattering approach, based on the formulation of a Riemann–Hilbert problem, for the initial value problem for the Camassa–Holm (CH) equation [3]

$$u_t - u_{txx} + 2\omega u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad -\infty < x < \infty,\ t > 0, \tag{1a}$$

$$u(x, 0) = u_0(x), \tag{1b}$$

where $\omega > 0$ is a parameter and $u_0(x)$ is assumed to be sufficiently smooth and with fast decay as $|x| \to \infty$. This approach allows studying the long time behavior of solutions to (1) with the help of the nonlinear steepest descent method of Deift and Zhou [8]. In this paper, we deal with the solitonic part of the asymptotics for $\omega > 0$ and relate it, for small ω, to trains of peakons, which are piecewise smooth soliton-type solutions to (1) with $\omega = 0$.

The CH equation is a model equation [3] describing the shallow-water approximation in inviscid hydrodynamics, where the constant ω is related to the critical shallow water wave speed. It is known (see, e.g., [4]) that if $m(x, 0) + \omega > 0$ for all x, where $m := u - u_{xx}$, then $m(x, t)$ exists for all $t > 0$; moreover, $m(x, t) + \omega > 0$, which justifies the equivalent form of the CH equation

$$\left(\sqrt{m + \omega}\right)_t = -\left(u\sqrt{m + \omega}\right)_x. \tag{2}$$

Starting from the standard Lax pair representation of the CH equation [3]

$$\psi_{xx} = \frac{1}{4}\psi + \lambda(m + \omega)\psi, \tag{3a}$$

$$\psi_t = \left(\frac{1}{2\lambda} - u\right)\psi_x + \frac{1}{2}u_x\psi, \tag{3b}$$

we develop a 'direct scattering–inverse scattering' formalism, in which the Lax pair is used in the form of a system of first order matrix-valued linear equations, appropriately chosen in order to facilitate the study of the analytic properties of solutions as functions of the spectral parameter. We show that:

(i) The solution of the initial value problem (1) can be obtained in a parametric form via the solution of the associated Riemann–Hilbert problem.

(ii) In the solitonic sector, $x > 2\omega t$, a solution of (1) develops into a (finite) train of (analytic) solitons.
(iii) As $\omega \to 0$, this soliton train approaches the train of peakons associated with the eigenvalues of the spectral problem (3a) with $\omega = 0$.

An alternative inverse scattering method based on an additive Riemann–Hilbert problem formulation for the associated eigenfunctions is presented in [5].

2. Eigenfunctions, spectral functions, and the Riemann–Hilbert problem

2.1. $\omega = 1$

We present the general formalism scaling out the parameter ω in the CH equation and thus assuming that $\omega = 1$ and $m + 1 > 0$. Let $u(x,t)$ be a solution of (1a) with $\omega = 1$ such that $u(x,t) \to 0$ as $|x| \to \infty$ for all t. Introducing

$$k^2 := -\lambda - \frac{1}{4}, \quad G_\infty(x,t,k) := \frac{1}{2}\begin{pmatrix} 1 & -\frac{1}{\mathrm{i}k} \\ 1 & \frac{1}{\mathrm{i}k} \end{pmatrix} \begin{pmatrix} (m+1)^{1/4} & 0 \\ 0 & (m+1)^{-1/4} \end{pmatrix}, \quad \tilde{\Phi} := G_\infty \begin{pmatrix} \psi \\ \psi_x \end{pmatrix}$$

we rewrite the Lax pair equations as

$$\tilde{\Phi}_x + \mathrm{i}k\sqrt{m+1}\,\sigma_3\tilde{\Phi} = U\tilde{\Phi}, \tag{4a}$$

$$\tilde{\Phi}_t + \mathrm{i}k\left(\frac{1}{2\lambda} - u\sqrt{m+1}\right)\sigma_3\tilde{\Phi} = V\tilde{\Phi}, \tag{4b}$$

where $\sigma_3 = \operatorname{diag}\{1,-1\}$, $U(x,t,k) \to 0$ and $V(x,t,k) \to 0$ as $|x| \to \infty$, and:

$$U(x,t,k) = \frac{1}{4}\frac{m_x}{m+1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{1}{8\mathrm{i}k}\frac{m}{\sqrt{m+1}}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix},$$

$$V(x,t,k) = -uU + \frac{\mathrm{i}k}{4\lambda}\sqrt{m+1}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} + \left(\frac{u}{4\mathrm{i}k} + \frac{\mathrm{i}k}{4\lambda}\right)\frac{1}{\sqrt{m+1}}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} + \frac{\mathrm{i}k}{2\lambda}\sigma_3.$$

Eqs. (4) suggest defining a scalar function $p(x,t,k)$ such that $p_x = \sqrt{m+1}$ and $p_t = \frac{1}{2\lambda} - u\sqrt{m+1}$; due to (2), this is possible, so we define p by

$$p(x,t,k) := x - \int_x^\infty \left(\sqrt{m(\xi,t)+1} - 1\right)\mathrm{d}\xi + \frac{t}{2\lambda(k)}. \tag{5}$$

Letting $\Phi(x,t,k) := \tilde{\Phi}\mathrm{e}^{\mathrm{i}kp(x,t,k)\sigma_3}$, (4) becomes

$$\Phi_x + \mathrm{i}kp_x[\sigma_3, \Phi] = U\Phi, \tag{6a}$$

$$\Phi_t + \mathrm{i}kp_t[\sigma_3, \Phi] = V\Phi, \tag{6b}$$

where $[a,b] := ab - ba$. The Lax pair in the form (6) allows defining the solutions Φ_+ and Φ_- ($\det \Phi_\pm \equiv 1$) with a well-controlled behavior in the k-plane via integral Volterra equations:

$$\Phi_\pm(x,t,k) = I + \int_{\pm\infty}^{x} \mathrm{e}^{-\mathrm{i}k\int_y^x \sqrt{m(\xi,t)+1}\,\mathrm{d}\xi\,\hat{\sigma}_3}(U\Phi_\pm)(y,t,k)\,\mathrm{d}y, \tag{7}$$

where I is the 2×2 identity matrix and $\mathrm{e}^{\hat{\sigma}_3}A := \mathrm{e}^{\operatorname{ad}\sigma_3}A = \operatorname{Ad}\mathrm{e}^{\sigma_3}\cdot A = \mathrm{e}^{\sigma_3}A\mathrm{e}^{-\sigma_3}$ for any 2×2 matrix A. As functions of k, $\Phi_\pm \equiv (\Phi_\pm^{(1)}\ \Phi_\pm^{(2)})$ are such that the vectors $\Phi_-^{(1)}$ and $\Phi_+^{(2)}$ are analytic in $\{k \mid \operatorname{Im} k > 0\}$ and $\Phi_+^{(1)}$ and $\Phi_-^{(2)}$ are analytic in $\{k \mid \operatorname{Im} k < 0\}$ (with possible pole, of a particular structure, at $k = 0$); moreover, as $k \to \infty$, $(\Phi_-^{(1)}\ \Phi_+^{(2)}) \to I$ and $(\Phi_+^{(1)}\ \Phi_-^{(2)}) \to I$.

For $k \in \mathbb{R}$, the eigenfunctions Φ_- and Φ_+ are related by the spectral matrix $s(k)$:

$$\Phi_+(x,t,k) = \Phi_-(x,t,k)\mathrm{e}^{-\mathrm{i}kp(x,t,k)\hat{\sigma}_3}s(k) \quad \text{with } s(k) = \begin{pmatrix} \overline{a(k)} & b(k) \\ \overline{b(k)} & a(k) \end{pmatrix}, \ k \in \mathbb{R}, \ k \neq 0, \tag{8}$$

where $\overline{a(k)} = a(-k)$ and $\overline{b(k)} = b(-k)$. Let $\{k_j\}_{j=1}^N$ be the set of zeros of $a(k)$. They are related to the eigenvalues $\{\lambda_j\}$ of (3a) with $m = m(x,0)$ by $\lambda_j = -k_j^2 - \frac{1}{4}$. Thus (see [4]), they are simple with $\frac{\mathrm{d}a}{\mathrm{d}k}(k_j) \in \mathrm{i}\mathbb{R}$, $N < \infty$, for all $1 \leqslant j \leqslant N$, $k_j = \mathrm{i}\nu_j$ with $0 < \nu_j < \frac{1}{2}$, and

$$\Phi_-^{(1)}(x,t,\mathrm{i}\nu_j) = \chi_j \mathrm{e}^{-2\nu_j p(x,t,\mathrm{i}\nu_j)} \Phi_+^{(2)}(x,t,\mathrm{i}\nu_j), \quad \chi_j \in \mathbb{R}. \tag{9}$$

The conjugating factors in (8) and (9) involve, on one hand, the spectral data $(r(k), \{\nu_j\}, \{\chi_j\})$ determined by the initial data $m(x,0)$ via the solution of the direct scattering problem, see (7)–(9), and, on the other hand, the exponential factors involving $p(x,t,k)$, which is defined by $m(x,t)$ for all t. Hence, in order to interpret (8)–(9) as a Riemann–Hilbert problem, the data for which are determined by the initial data $u(x,0)$ only, a rescaling is proposed following the structure of p:

$$y(x,t) = x - \int_x^\infty \left(\sqrt{m(\xi,t)+1} - 1\right) \mathrm{d}\xi. \tag{10}$$

Notice that by (2) we have $\frac{\partial x}{\partial t}(y,t) = u(x,t)$.

In order to relate u to $\Phi_\pm$, we use a particular feature of the Lax pair for the Camassa–Holm equation: the x-equation (3a) becomes trivial (independent of m) for $\lambda = 0$, which corresponds to $k = \pm\frac{\mathrm{i}}{2}$. In terms of $\Phi_\pm$, this property reads as

$$\left(\Phi_-^{(1)}\ \Phi_+^{(2)}\right)\left(x,t,\frac{\mathrm{i}}{2}\right) = F(x,t)\,\mathrm{diag}\left\{\mathrm{e}^{-\frac{1}{2}\int_x^\infty(\sqrt{m(\xi,t)+1}-1)\,\mathrm{d}\xi},\ \mathrm{e}^{-\frac{1}{2}\int_{-\infty}^x(\sqrt{m(\xi,t)+1}-1)\,\mathrm{d}\xi}\right\}$$

and $a(\frac{\mathrm{i}}{2}) = \exp\{-\frac{1}{2}\int_{-\infty}^\infty(\sqrt{m(\xi,t)+1} - 1)\,\mathrm{d}\xi\}$, where $F = \frac{1}{2}\left(\begin{smallmatrix} q+q^{-1} & q-q^{-1} \\ q-q^{-1} & q+q^{-1}\end{smallmatrix}\right)$ with $q(x,t) = (m(x,t)+1)^{1/4}$. Then the scales x and y can be related in terms of $\Phi_\pm(x,t,\frac{\mathrm{i}}{2})$: letting $\tilde{\mu}_1(x,t) := (\Phi_{-11}(x,t,\frac{\mathrm{i}}{2}) + \Phi_{-21}(x,t,\frac{\mathrm{i}}{2}))/a(\frac{\mathrm{i}}{2})$ and $\tilde{\mu}_2(x,t) := \Phi_{+12}(x,t,\frac{\mathrm{i}}{2}) + \Phi_{+22}(x,t,\frac{\mathrm{i}}{2})$ we have

$$\frac{\tilde{\mu}_1(x,t)}{\tilde{\mu}_2(x,t)} = \mathrm{e}^{\int_x^\infty(\sqrt{m(\xi,t)+1}-1)\,\mathrm{d}\xi} = \mathrm{e}^{x-y(x,t)}. \tag{11}$$

Now the scattering problem (8), (9) can be rewritten as a vector RH problem parametrized by (y,t):

Riemann–Hilbert problem 2.1. *Given $r(k) = b(k)/\overline{a(k)}$ for $k \in \mathbb{R}$, $\{\nu_j\}_{j=1}^N$ $(0 < \nu_j < \frac{1}{2})$, and $\{\gamma_j\}_{j=1}^N$ $(\gamma_j > 0)$, find a row function $\mu(y,t,k) = (\mu_1,\mu_2)(y,t,k)$ such that*:

(a) *$\mu(\cdot,\cdot,k)$ is analytic in $\{k \mid \operatorname{Im} k > 0\}$ and in $\{k \mid \operatorname{Im} k < 0\}$*;
(b) $\overline{\mu_1(\cdot,\cdot,\bar{k})} = \mu_1(\cdot,\cdot,-k) = \mu_2(\cdot,\cdot,k)$;
(c) *$\mu(y,t,k) \to (1,1)$ as $k \to \infty$*;
(d) *we have the jump relation*

$$\mu_-(y,t,\zeta) = \mu_+(y,t,\zeta)J(y,t,\zeta), \quad \zeta \in \mathbb{R},\ \text{with } J(y,t,k) = \mathrm{e}^{-\mathrm{i}k(y-\frac{2}{1+4k^2}t)\hat{\sigma}_3} J_0(k), \tag{12}$$

where $J_0(k) = \left(\begin{smallmatrix} 1 & -r(k) \\ r(k) & 1-|r(k)|^2\end{smallmatrix}\right)$;

(e) $\mathrm{Res}_{k=\mathrm{i}\nu_j}\,\mu_1(y,t,k) = \mathrm{i}\gamma_j \mathrm{e}^{-2\nu_j(y-\frac{2}{1-4\nu_j^2}t)} \mu_2(y,t,\nu_j)$ *where* $\mathrm{i}\gamma_j = \frac{\chi_j}{(\mathrm{d}a/\mathrm{d}k)(\mathrm{i}\nu_j)}$, *and, by symmetry, the corresponding relation at $k = -\mathrm{i}\nu_j$.*

This RH problem has the same structure as that for the Korteweg–de Vries equation (except for the k-dependence of the velocity in the phase factors), which implies that there exists a vanishing lemma [1] ensuring that the RH problem has a unique solution $\mu(y,t,k)$ for all $y \in (-\infty,\infty)$ and $t > 0$. Evaluating this solution at $k = \mathrm{i}/2$, the solution $u(x,t)$ of the initial value problem for the Camassa–Holm equation is obtained in the parametric form (cf. (11)):

$$u(x,t) = \left(\frac{\partial}{\partial t}\ln\frac{\mu_1}{\mu_2}\left(y,t,\frac{\mathrm{i}}{2}\right)\right), \qquad x(y,t) = y + \ln\frac{\mu_1(y,t,\frac{\mathrm{i}}{2})}{\mu_2(y,t,\frac{\mathrm{i}}{2})}. \tag{13}$$

Eqs. (13) have the same structure as the parametric formulas representing pure multisoliton solutions [9] in terms of the ratio of two determinants. Therefore, the multisoliton solutions [9] are "embedded" into our scheme for the solution of the initial value problem: they correspond to reflectionless ($r(k) \equiv 0$) initial data, for which the solution of the RH problem is reduced to solving linear algebraic equations.

If $N = 1$, $k_1 = \mathrm{i}\nu_1 =: \mathrm{i}\nu$ then

$$\big(\mu_1(y,t,k), \mu_2(y,t,k)\big) = \left(\frac{k - B(y,t)}{k - \mathrm{i}\nu}, \frac{k + B(y,t)}{k + \mathrm{i}\nu}\right) \quad \text{with } B = \mathrm{i}\nu\frac{1-g}{1+g},$$

where $g(y,t) = \exp\{-2\nu(y - \frac{2}{1-4\nu^2}t - y_0)\}$ with $y_0 = \frac{1}{2\nu}\ln\frac{\gamma}{2\nu}$. Introducing $v := \frac{2}{1-4\nu^2}$ and $\phi := -2\nu(y - vt - y_0)$, the 1-soliton is given parametrically by

$$u(y,t) = \frac{16\nu^2}{1-4\nu^2}\frac{1}{1+4\nu^2+(1-4\nu^2)\cosh\phi}, \qquad x(y,t) = y + \ln\frac{1+g(1+2\nu)/(1-2\nu)}{1+g(1-2\nu)/(1+2\nu)}. \tag{14}$$

The representation of a solution of a nonlinear equation in terms of the solution of the associated Riemann–Hilbert problem allows studying its long time behavior by utilizing the nonlinear steepest descent method by Deift and Zhou [8]. In particular, in the soliton region $y > 2t$, the oscillating factor $\mathrm{e}^{2\mathrm{i}t\theta}$ in (12) with $\theta(y,t,k) = \frac{y}{t}k - \frac{2k}{1+4k^2}$ is such that $\operatorname{Im}\theta(k)$ changes its sign when k crosses the real axis along all $\mathbb{R}$, which implies that the factorization $J = \left(\begin{smallmatrix}1 & 0\\ \overline{r(k)}\mathrm{e}^{2\mathrm{i}t\theta} & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & -r(k)\mathrm{e}^{-2\mathrm{i}t\theta}\\ 0 & 1\end{smallmatrix}\right)$ can be used for all $k \in \mathbb{R}$ in order to deform the RH problem to a form with the jump matrix exponentially approaching I as $t \to \infty$ (see, e.g., [8]). This implies that the solution of the 2×2 *regular* RH problem with the jump matrix as in (12) also approaches I for all k with $|\operatorname{Im} k| > \varepsilon > 0$; in turn, this (together with $x - y = \mathrm{o}(1)$ as $y \to +\infty$) yields

$$u(x,t) = u_s(x,t) + \mathrm{o}(1), \quad t \to \infty,$$

where $u_s(x,t)$ is the pure N-soliton solution of the CH equation [9], which in turn develops, for large t, into a superposition of 1-solitons of type (14).

2.2. $\omega > 0$

The solution algorithm presented for the case $\omega = 1$ works for any $\omega > 0$ if we replace m by $\frac{m}{\omega}$ and λ by $\lambda\omega$. Accordingly, $p(x,t,k)$ and $y(x,t)$ become ω-dependent:

$$p(x,t,k) = y(x,t) - \frac{2\omega}{1+4k^2}t, \qquad y(x,t) = x - \int_x^\infty\left(\sqrt{\frac{m(\xi,t)}{\omega}+1} - 1\right)\mathrm{d}\xi.$$

3. The limit transition $\omega \to 0$

Now consider a family of initial value problems (1) parametrized by ω, where the initial function $u_0(x)$ in (1b) is always the same. The results of the preceding section show that for every fixed ω, in the domain $x > 2\omega t$ the observer will see a finite train of solitons, whose parameters are determined by $\{\nu_j^\omega\}_{j=1}^{N_\omega}$ and $\{\gamma_j^\omega\}_{j=1}^{N_\omega}$. On the other hand, it has been observed [11,9] that if the parameters of an ω-soliton are changing appropriately with ω when $\omega \to 0$, then this soliton approaches a peakon, which is a piecewise smooth (weak), stable (cf. [2,7]) solution of (1a) with $\omega = 0$: $u^0(x,t) = v^0\mathrm{e}^{-|x-v^0t-x^0|}$ with $v^0 > 0$.

Theorem 3.1. *Let $u_0(x)$ be a smooth, rapidly decreasing, as $|x| \to \infty$, function such that $m_0(x) := u_{0xx}(x) - u_0(x) > 0$ for all $x \in \mathbb{R}$. Let $\{\lambda_j^0\}_{j=1}^\infty$ be the eigenvalues of the spectral problem* (3a) *with $\omega = 0$ and $m = m_0$. For $\omega > 0$, let $u^\omega(x,t)$ be a solution to the initial value problem for the Camassa–Holm equation* (1).

Fix $C > 0$, $\delta > 0$, and $\varepsilon > 0$. Let $\{\lambda_j^0\}_{j=1}^{N(C)}$ be those λ_j^0 satisfying $0 > \lambda_1^0 > \cdots > \lambda_{N(C)}^0 > -\frac{1}{2C}$.

Then there exists $\tilde\omega = \tilde\omega(C,\delta,\varepsilon)$ such that for all $0 < \omega < \tilde\omega$ the asymptotics of $u^\omega(x,t)$ in the domain $x > Ct$ is given by $N(C)$ one-solitons of type (15) *with velocities and forms close to those of the corresponding peakons*:

$$u^\omega(x,t) = U_j^\omega(X) + \mathrm{o}(1) \quad \textit{as } t \to \infty \quad \textit{and} \quad |X| = \mathrm{O}(1) \quad \textit{with } X = x - v_j^\omega t - x_{0j}^\omega,$$

where $o(1)$ *depends on* ω, $|v_j^\omega - v_j^0| < \varepsilon$ *with* $v_j^0 = -1/(2\lambda_j^0)$, $|x_{0j}^\omega - x_{0j}^0| < \varepsilon$ *with* $x_{0j}^0 = \ln\{\frac{\Gamma_j^0}{-\lambda_j^0}\prod_{l=1}^{j-1}(\lambda_j^0/\lambda_l^0 - 1)^2\}$, *and* $|U_j^\omega(X) - v_j^0 e^{-|X|}| < \varepsilon$ *for* $|X| > \delta$.

The proof is based on the following observation: let $\Psi_\pm^\omega(x, \lambda_j^\omega)$, $\omega \geqslant 0$ be the eigenfunctions of (3a) associated with the eigenvalues λ_j^ω and normalized by $\Psi_\pm^\omega(x, \lambda_j^\omega) \to e^{\mp\nu_j^\omega x}$ as $x \to \pm\infty$ with $\nu_j^\omega = \sqrt{\omega\lambda_j^\omega + \frac{1}{4}}$ for $\omega > 0$ and $\Psi_\pm^0(x, \lambda_j^0) \sim e^{\mp x/2}$ as $x \to \pm\infty$ for $\omega = 0$. Then, passing from (3a) to the spectral problem $K^*(m + \omega)Kf = \frac{1}{\lambda} f$ with $(Kg)(x) = \int_x^\infty e^{\frac{x-y}{2}} g(y)\,dy$ [10] and considering ω as the perturbation parameter, it can be seen that as $\omega \to 0$, we have $N_\omega \to \infty$, $\lambda_j^\omega \to \lambda_j^0$, and $\Psi_\pm^\omega(x, \lambda_j^\omega) \to \Psi_\pm^\omega(x, \lambda_j^0)$ in $L^2(-\infty, \infty)$, $j = 1, 2, \ldots$, in the sense that new ω-eigenvalues are "escaping" from the continuous spectrum whereas the already existing ω-eigenvalues and the associated ω-eigenfunctions are approaching respectively the corresponding eigenvalues and eigenfunctions of (3a) with $\omega = 0$ [6]. In turn, this implies that $2\nu_j^\omega \sim 1 + 2\omega\lambda_j^0$ and $\gamma_j^\omega \sim \omega\Gamma_j^0$ with $(\Gamma_j^0)^{-1} = \int_{-\infty}^\infty m_0(x)|\Psi_+^0(x, \lambda_j^0)|^2\,dx$. Hence, the velocity v^ω and the phase x_0^ω of each ω-soliton

$$u(x,t) = U\big(Y(X)\big)\big|_{X = x - v^\omega t - x_0^\omega} \equiv U^\omega(X)|_{X = x - v^\omega t - x_0^\omega}, \tag{15}$$

where

$$U(Y) = \frac{16\omega(\nu^\omega)^2}{1 - 4(\nu^\omega)^2}\frac{1}{1 + 4(\nu^\omega)^2 + (1 - 4(\nu^\omega)^2)\cosh(2\nu^\omega Y)}, \quad X(Y) = Y + \ln\frac{1 - 2\nu^\omega + (1 + 2\nu^\omega)e^{-2\nu^\omega Y}}{1 + 2\nu^\omega + (1 - 2\nu^\omega)e^{-2\nu^\omega Y}},$$

approach finite limits as $\omega \to 0$: $v^\omega = \frac{2\omega}{1 - 4(\nu^\omega)^2} \to -\frac{1}{2\lambda^0}$ and $x_0^\omega = \frac{1}{2\nu^\omega}\ln\frac{\gamma^\omega}{2\nu^\omega} + \ln\frac{1+2\nu^\omega}{1-2\nu^\omega} \to \ln\frac{\Gamma^0}{-\lambda^0}$. The form of the limiting phase x_{0j}^0 follows taking into account the phase shift when passing from a multisoliton solution to a superposition of one-solitons [9].

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