



Numerical Analysis

Influence coefficients for variational integral equations

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Received 17 March 2005; accepted after revision 25 March 2006

Available online 17 October 2006

Presented by Philippe G. Ciarlet

Abstract

We compute exact formulas for the influence coefficients deriving from the finite element discretization of integral equation methods. We consider the case of the Newtonian potential and plane triangles of the lower degree. **To cite this article:** *M. Lenoir, C. R. Acad. Sci. Paris, Ser. I 343 (2006)*.

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Résumé

Coefficients d'influence pour des méthodes d'équations intégrales. Nous établissons des formules exactes pour les coefficients d'influence issus de la discrétisation par éléments finis des méthodes d'équations intégrales, dans le cas du potentiel Newtonien et de triangles plans du plus bas degré. **Pour citer cet article :** *M. Lenoir, C. R. Acad. Sci. Paris, Ser. I 343 (2006)*.

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1. Introduction

The present Note is a contribution to the computation of near-diagonal terms for the discretization of variational integral equations [2]. The method of calculation relies upon a quadrature formula for homogeneous functions which allows a recursive reduction of the dimension of the integration domain, provided there exists a convenient origin of the axes. Our formulas give the (almost) singular part of the integrals and must be completed by numerical integration techniques for the remainder. They must be considered as an alternative to purely numerical techniques [1]. When the triangles do not meet or share only one vertex, parallelism may occur between some sides or between the planes of the triangles, resulting in cancellation between singular terms. The number of such terms being fixed, there is no limit to the precision which can be reached, provided stable formulas have been obtained for the limit cases. They have been obtained for the case of parallel sides and are under examination for the case of parallel planes, they will be accounted for in a forthcoming paper together with more detailed calculations.

By S and T we denote two triangles, with vertices a_i and b_j . We put $a_i^+ = a_{i+2}$, $a_i^- = a_{i+1}$, and denote by α_i the side $[a_i^-, a_i^+]$. Any side of S is denoted by $\alpha = [a^-, a^+]$, and similarly for T , with $\beta = [b^-, b^+]$. The projection of a^ℓ on β is denoted by q^ℓ and of b^k on α , by p^k ; any point of β is denoted by b and its projection on α by p . The abscissas of a^ℓ and p along α with origin c_α are s^ℓ and σ . Let $G(x, y) = \|x - y\|^{-1}$, we shall be concerned with

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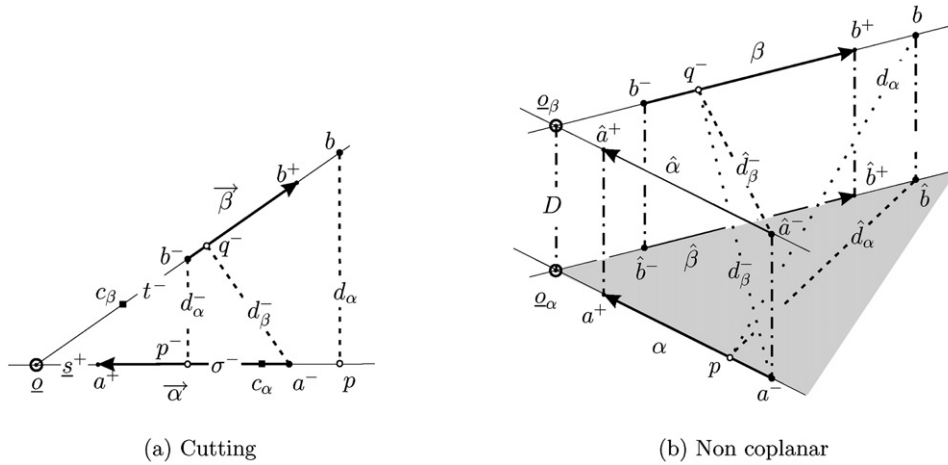


Fig. 1. Influence between segments.

$$\mathcal{T}(S, T) = \int_{S \times T} G(x, y) dx dy, \quad \mathcal{S}(\beta, S) = \int_{S \times \beta} G(x, y) dx dy, \quad \mathcal{Q}(\alpha, \beta) = \int_{\alpha \times \beta} G(x, y) dx dy, \quad \text{and} \quad (1)$$

$$\mathcal{P}(b, S) = \int_S G(x, b) dx, \quad \mathcal{R}(b, \alpha) = \int_\alpha G(x, b) dx = \sum_{\ell=\pm} \ell \text{Arg sh } \frac{\rho^\ell}{d_\alpha}, \quad \rho^\ell = s^\ell - \sigma, \quad d_\alpha = \|p - b\|. \quad (2)$$

2. Influence between segments

(i) Assume first that the supports of α and β intersect, Fig. 1(a); let o the intersection and s^ℓ and t^k the abscissas of a^ℓ and b^k with respect to o . Then $G(\underline{x}, \underline{y})$ is homogeneous w.r.t. (s, t) , i.e. on $\alpha \times \beta$, and as $\mathcal{Q}(\alpha, \beta) = \int_{\alpha \times \beta} G(\underline{x}, \underline{y}) d\underline{x} d\underline{y}$, we may call upon formula (16) with $n = 2$ et $\Omega = \alpha \times \beta$, which gives

$$\mathcal{Q}(\alpha, \beta) = \sum_{k=\pm} k \{ s^k \mathcal{R}(a^k, \beta) + t^k \mathcal{R}(b^k, \alpha) \}. \quad (3)$$

(ii) If α and β do not belong to the same plane, Fig. 1(b); let o_α and o_β belong to the supports of α and β , such that $o_\beta - o_\alpha$ is orthogonal to α and β . By $\hat{\beta}$ we denote the projection of β on plane \mathfrak{P} , containing α and parallel to β . Let \hat{b} the projection of b on $\hat{\beta}$, $\hat{d}_\alpha = \|\hat{b} - p\|$ and $D = \|o_\beta - o_\alpha\|$; then by (15), $\mathcal{Q}(\alpha, \beta) = D \sum_{\ell=\pm} \ell s^\ell \int_\beta \mathcal{T}_D(\hat{a}^\ell - y) dy + D \sum_{k=\pm} k t^k \int_\alpha \mathcal{T}_D(x - \hat{b}^k) dx$, i.e.

$$\mathcal{Q}(\alpha, \beta) = \sum_{\ell=\pm} \ell s^\ell \mathcal{U}_D(\hat{a}^\ell, \beta) + \sum_{k=\pm} k t^k \mathcal{U}_D(\hat{b}^k, \alpha), \quad \text{where} \quad (4)$$

$$\mathcal{U}_D(\hat{b}, \alpha) = \sum_{\ell=\pm} \ell \left\{ \text{Arg sh } \frac{\rho^\ell}{d_\alpha} - \frac{D}{\hat{d}_\alpha} \text{Arc tg } \frac{\rho^\ell}{\hat{d}_\alpha} + \frac{D}{2\hat{d}_\alpha} \text{Arc tg } \frac{2\rho^\ell \hat{d}_\alpha D \|b - a^\ell\|}{(\rho^\ell)^2 (|\hat{d}_\alpha|^2 - D^2) + |d_\alpha|^2 |\hat{d}_\alpha|^2} \right\}. \quad (5)$$

3. Potential arising from a triangle

(i) If b belongs to the plane of S , we choose $\bar{o} = b$ as origin, we put $\delta_{\alpha_i}(b) = (x - b | \vec{\mu}_i)$, where $x \in \alpha_i$ and $\vec{\mu}_i$ is the exterior normal to S along α_i . Then by (16), with $n = 2$, we obtain

$$\mathcal{P}(b, S) = \sum_{i=1,3} \delta_{\alpha_i}(b) \mathcal{R}(b, \alpha_i). \quad (6)$$

(ii) If b does not, we denote by \check{b} the projection of b on the plane of S , and we choose $\bar{o} = \check{b}$. Then by (15) we obtain $\mathcal{P}(b, S) = \Delta_S(b) \int_{\partial \bar{S}} (\bar{x} | \bar{\mu}) \mathcal{T}_{\Delta_S(b)}(\bar{x}) d\sigma(\bar{x})$, where $\Delta_S(b) = \|b - \check{b}\|$, and by (15),

$$\mathcal{P}(b, S) = \sum_{i=1,3} \delta_{\alpha_i}(\check{b}) \mathcal{U}_{\Delta_S(b)}(\check{b}, \alpha_i). \tag{7}$$

4. Influence between a segment and a triangle

(i) Let first the segment be some side of the triangle, i.e. $\beta = \alpha_i$, and choose as origin $\bar{o} = a_i^- = a_{i+1}$. By (16), with $n = 3$, we obtain $\mathcal{S}(S, \alpha_i) = \frac{1}{2} |\alpha_i| \mathcal{P}(a_{i+2}, S) + \frac{1}{2} \delta_{\alpha_{i+1}}(a_{i+1}) \mathcal{Q}(\alpha_i, \alpha_{i+1})$, where $\mathcal{P}(a_{i+2}, S)$ and $\mathcal{Q}(\alpha_i, \alpha_{i+1})$ follow respectively from (6) and (3). As a consequence

$$\mathcal{S}(\alpha_i, S) = |S| \mathcal{R}(a_i, \alpha_i) + \frac{|\alpha_i|}{2} \sum_{j \neq i} \delta_{\alpha_j}(a_j) \mathcal{R}(a_j, \alpha_j), \tag{8}$$

where $|S|$ is the surface of the triangle.

(ii) If the triangle and the support of the segment meet at \bar{o}_S^β , then

$$\mathcal{S}(\beta, S) = \frac{1}{2} \sum_{k=\pm} k \bar{t}_S^k \mathcal{P}(b^k, S) + \frac{1}{2} \sum_{i=1,3} \delta_{\alpha_i}(\bar{o}_S^\beta) \mathcal{Q}(\beta, \alpha_i), \tag{9}$$

which greatly simplifies if one end of the segment is a vertex of the triangle.

5. Influence between two triangles

(i) In the case of a self-influence coefficient. As origin, we choose $\bar{o} = a_i$, and by formula (16), with $n = 4$, we obtain $\mathcal{T}(S, S) = \frac{2}{3} d_{\alpha_i}^i \mathcal{S}(\alpha_i, S)$, then with $d_{\alpha_i}^i$ the distance between a_i and α_i , from (8) we get

$$\mathcal{T}(S, S) = \frac{2}{3} |S| \sum_{i=1,3} d_{\alpha_i}^i \mathcal{R}(a_i, \alpha_i). \tag{10}$$

(ii) If the triangles have some common side, i.e. $\alpha_i = \beta_j$. Assume that $\bar{\alpha}_i = \bar{\beta}_j$, and put $\bar{o} = a_i^- = b_j^-$; it follows that $\mathcal{T}(S, T) = \frac{1}{3} d_{\alpha_{i+1}}^{i+1} \mathcal{S}(\alpha_{i+1}, T) + \frac{1}{3} d_{\beta_{j+1}}^{j+1} \mathcal{S}(\beta_{j+1}, S)$, and by (9)

$$\mathcal{T}(S, T) = \frac{1}{3} |S| \mathcal{P}(a_i, T) + \frac{1}{3} |T| \mathcal{P}(b_j, S) + \frac{1}{6} d_{\alpha_{i+1}}^{i+1} d_{\alpha_{i+2}}^{i+2} \{ \mathcal{Q}(\alpha_{i+1}, \beta_{j+2}) + \mathcal{Q}(\beta_{j+1}, \alpha_{i+2}) \}. \tag{11}$$

(iii) If the triangles have some common vertex, i.e. $a_i = b_j = \bar{o}$, one has

$$\mathcal{T}(S, T) = \frac{1}{3} d_{\alpha_i}^i \mathcal{S}(\alpha_i, T) + \frac{1}{3} d_{\beta_j}^j \mathcal{S}(\beta_j, S). \tag{12}$$

Moreover, if α_i (resp. β_j) is not parallel to T (resp. S), then $\mathcal{S}(\alpha_i, T)$ and $\mathcal{S}(\beta_j, S)$ follow from (9).

(iv) If some point, say \bar{o} , belongs to the planes of the two triangles, we denote by $d_{\alpha_i}(\bar{o})$ the distance between \bar{o} and α_i , and we obtain

$$\mathcal{T}(S, T) = \frac{1}{3} \sum_{i=1,3} d_{\alpha_i}(\bar{o}) \mathcal{S}(\alpha_i, T) + \frac{1}{3} \sum_{j=1,3} d_{\beta_j}(\bar{o}) \mathcal{S}(\beta_j, S). \tag{13}$$

Moreover, if no side of any triangle is parallel to the other one, let $\Theta_{ij} = d_{\alpha_i}(\bar{o}) d_{\beta_j}(\bar{o}_T^i) + d_{\beta_j}(\bar{o}) d_{\alpha_i}(\bar{o}_S^j)$, where \bar{o}_T^i and \bar{o}_S^j are the respective intersections of the support of α_i (resp. β_j) and of the plane of T (resp. S), then

$$\mathcal{T}(S, T) = \frac{1}{6} \sum_{i=1,3} \left\{ d_{\alpha_i}(\bar{o}) \sum_{\ell=\pm} \ell \mathcal{L}_i^\ell \mathcal{P}(a_i^\ell, T) + d_{\beta_i}(\bar{o}) \sum_{k=\pm} k \mathcal{L}_i^k \mathcal{P}(b_i^k, S) + \sum_{j=1,3} \Theta_{ij} \mathcal{Q}(\alpha_i, \beta_j) \right\}. \tag{14}$$

6. Integration of homogeneous functions

By Ω we denote a bounded domain of \mathbb{R}^n , by $w = (z, v)$ some element of $\mathbb{R} \times \Omega$ and by f some positively homogeneous function of degree -1 . Let ν the exterior normal to Ω and $I(z)/z^{n-1} \rightarrow 0$ as $z \rightarrow \infty \operatorname{sgn} z$, then (see [3])

$$I(z) = \int_{\Omega} f(z, v) \, dv = z^{n-1} \int_{\partial\Omega} (v|\nu) \mathcal{T}_z(v) \, d\sigma(v), \quad \text{where } \mathcal{T}_z(v) = \int_z^{\infty \operatorname{sgn} z} \frac{f(t, v)}{t^n} \, dt. \quad (15)$$

In the case where f does not depend on z , one obtains

$$(n-1)I = \int_{\partial\Omega} (v|\nu) f(v) \, d\sigma(v). \quad (16)$$

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