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Algebra

A universal deformation ring which is not a complete intersection ring

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Abstract

Bleher and Chinburg recently used modular representation theory to produce an example of a linear representation of a finite group whose universal deformation ring is not a complete intersection ring. We prove this by using only elementary cohomological obstruction calculus. ***To cite this article: J. Byszewski, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Un anneau de déformation qui n'est pas d'intersection complète. Bleher et Chinburg ont récemment utilisé la théorie des réprésentation modulaires pour construire une représentation d'un groupe fini ayant un anneau de déformations universel qui n'est pas d'intersection complète. On redémontre ce résultat en n'utilisant que la théorie cohomologique des obstructions. ***Pour citer cet article : J. Byszewski, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Let k be a perfect field of characteristic 2 and let W denote the ring of Witt vectors of k. Consider the (unique) indecomposable representation of S_4 of dimension 2. In Theorem 3.1 we prove that its universal deformation ring is $W[[t]]/(t^2, 2t)$. This gives an example of a universal deformation ring which is not a complete intersection ring. The proof of Theorem 3.1 uses only cohomological obstruction calculus, the crucial part being an explicit computation of group cohomology of the adjoint representation (cf. Lemma 2.4). Another proof using modular representation theory is due to Bleher and Chinburg [1].

1. A representation of S_4 in characteristic 2

Let V be a representation of S_4 defined by the following group homomorphisms

$$S_4 \to S_3 \xrightarrow{\sim} \mathrm{GL}_2(\mathbf{F}_2) \hookrightarrow \mathrm{GL}_2(k) \xrightarrow{\sim} \mathrm{Aut}_k(V). \tag{1}$$

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The map $S_4 \to S_3$ is a quotient map and the kernel is the (unique) normal subgroup K of S_4 of order 4. It is isomorphic to the Klein four group. For computational purposes we give an explicit description of this representation. We write S_4 in terms of generators and relations

$$S_4 = \left\langle u, v, r, s \;\middle|\; \begin{array}{c} u^2 = v^2 = r^3 = s^2 = 1, uv = vu, srs = r^{-1} \\ sus = v, svs = u, rur^{-1} = v, rvr^{-1} = uv \end{array} \right\rangle. \tag{2}$$

Whenever we wish to interpret elements of S_4 as permutations on four letters, we choose the following identifications: $u = (12)(34)$, $v = (14)(23)$, $r = (123)$, $s = (13)$. The action of S_4 on V can be given by $\tau : S_4 \to \mathrm{GL}_2(k)$ with

$$\tau(u) = \tau(v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau(r) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \tau(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

2. Computing cohomology groups

Let M denote the adjoint representation of V, i.e., $M = M_2(k)$ with the action of S_4 given by conjugation $A \mapsto \tau(g)A\tau(g)^{-1}$. One of our aims is to compute the group cohomology $\mathrm{H}^1(S_4, M)$ and $\mathrm{H}^2(S_4, M)$, the first one being the tangent space to the deformation functor associated to V, the second one containing the obstructions. We identify S_3 with a subgroup of S_4 generated by r and s. Then V and M can also be regarded as S_3-modules.

Lemma 2.1. $\mathrm{H}^p(S_3, M) = 0$ *for* $p \geqslant 1$.

Proof. One can regard S_3 as a split extension of $\mathbf{Z}/2\mathbf{Z}$ by $\mathbf{Z}/3\mathbf{Z}$. The cohomology groups $\mathrm{H}^p(\mathbf{Z}/3\mathbf{Z}, M)$ are annihilated by both two and three, hence are trivial. Thus the short exact sequence

$$1 \to \mathbf{Z}/3\mathbf{Z} \to S_3 \to \mathbf{Z}/2\mathbf{Z} \to 1$$

induces inflation-restriction sequences (cf. [4, Chapter 7]) in every dimension $p \geqslant 1$

$$0 \to \mathrm{H}^p\big(\mathbf{Z}/2\mathbf{Z}, M^{\mathbf{Z}/3\mathbf{Z}}\big) \xrightarrow{\sim} \mathrm{H}^p(S_3, M) \xrightarrow{\mathrm{Res}} 0.$$

The action of $\mathbf{Z}/3\mathbf{Z}$ on M is given explicitly by (3), and therefore the computation of $M^{\mathbf{Z}/3\mathbf{Z}}$ amounts to solving a system of linear equations. Doing so yields $M^{\mathbf{Z}/3\mathbf{Z}} = \{\left(\begin{smallmatrix} a & b \\ b & a+b \end{smallmatrix}\right)\}$. The action of $\mathbf{Z}/2\mathbf{Z}$ on $M^{\mathbf{Z}/3\mathbf{Z}}$ is given by $\left(\begin{smallmatrix} a & b \\ b & a+b \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a+b & b \\ b & a \end{smallmatrix}\right)$. An easy computation using an explicit description of cohomology of cyclic groups yields $\mathrm{H}^p(\mathbf{Z}/2\mathbf{Z}, M^{\mathbf{Z}/3\mathbf{Z}}) = 0$ for $p \geqslant 1$, which finishes the proof. □

We proceed to calculate $\mathrm{H}^1(S_4, M)$ and $\mathrm{H}^2(S_4, M)$. The group S_3 acts on $\mathrm{H}^q(K, M)$ by

$$(g\eta)(s_1, \ldots, s_q) = g\big(\eta(g^{-1}s_1g, \ldots, g^{-1}s_qg)\big),$$

where η is a q-cocycle representing a cohomology class in $\mathrm{H}^q(K, M)$. We consider the cohomology groups $\mathrm{H}^p(S_3, \mathrm{H}^q(K, M))$. The short exact sequence $1 \to K \to S_4 \to S_3 \to 1$ induces a Hochschild–Serre spectral sequence $\mathrm{H}^p(S_3, \mathrm{H}^q(K, M)) \Rightarrow \mathrm{H}^{p+q}(S_4, M)$. Its 0th row is $\mathrm{H}^p(S_3, \mathrm{H}^0(K, M)) = \mathrm{H}^p(S_3, M) = 0$ for $p \geqslant 1$ by Lemma 2.1. The group $\mathrm{H}^1(K, M)$ is just the group $\mathrm{Hom}(K, M)$ with the action of S_3 given by $(g\varphi)(x) = g\varphi(g^{-1}xg)$. The action of S_3 by conjugation on K is given by $s^{-1}us = v$, $s^{-1}vs = u$, $s^{-1}uvs = uv$; $r^{-1}ur = uv$, $r^{-1}vr = u$, $r^{-1}uvr = v$. Reasoning as in Lemma 2.1, we obtain inflation maps inducing isomorphisms

$$\mathrm{H}^p\big(\mathbf{Z}/2\mathbf{Z}, \mathrm{H}^1(K, M)^{\mathbf{Z}/3\mathbf{Z}}\big) \xrightarrow{\sim} \mathrm{H}^p\big(S_3, \mathrm{H}^1(K, M)\big), \quad p \geqslant 1. \tag{4}$$

Lemma 2.2. $\dim_k \mathrm{H}^1(K, M)^{S_3} = 1$, $\mathrm{H}^p(S_3, \mathrm{H}^1(K, M)) = 0$ *for* $p \geqslant 1$.

Proof. This computation again amounts to solving linear equations. A homomorphism $\varphi : K \to M$ is invariant under the action of $\mathbf{Z}/3\mathbf{Z}$ if and only if $r\varphi = \varphi$. This is equivalent $\varphi(u) = (r\varphi)(u) = r\varphi(uv)$ and $\varphi(v) = (r\varphi)(v) = r\varphi(u)$. One easily checks that this is equivalent to $\varphi(u) = \left(\begin{smallmatrix} a & b \\ a+b & a \end{smallmatrix}\right)$, $\varphi(v) = \left(\begin{smallmatrix} b & a+b \\ a & b \end{smallmatrix}\right)$ for some $a, b \in k$. The action of $\mathbf{Z}/2\mathbf{Z}$ on $\mathrm{H}^1(K, M)^{\mathbf{Z}/3\mathbf{Z}}$ is given by $(s\varphi)(u) = s\varphi(v) = \left(\begin{smallmatrix} b & a \\ a+b & b \end{smallmatrix}\right)$, $(s\varphi)(v) = s\varphi(u) = \left(\begin{smallmatrix} a & a+b \\ b & a \end{smallmatrix}\right)$. Direct computation shows that $\mathrm{H}^p(\mathbf{Z}/2\mathbf{Z}, \mathrm{H}^1(K, M)^{\mathbf{Z}/3\mathbf{Z}}) = 0$ for $p \geqslant 1$ and that φ is invariant under the action of $\mathbf{Z}/2\mathbf{Z}$ if and only if $a = b$. Hence $\mathrm{H}^1(K, M)^{S_3}$ is one dimensional. The claim follows from isomorphisms (4). □

The spectral sequence takes on the following form:

$$\begin{array}{llll} \boxed{\mathrm{H}^0(S_3, \mathrm{H}^2(K, M))} & \mathrm{H}^1(S_3, \mathrm{H}^2(K, M)) & \mathrm{H}^2(S_3, \mathrm{H}^2(K, M)) & \ldots \\ \boxed{\mathrm{H}^0(S_3, \mathrm{H}^1(K, M))} & \boxed{0} & \boxed{0} & \boxed{\ldots} \\ \boxed{\mathrm{H}^0(S_3, \mathrm{H}^0(K, M))} & \boxed{0} & \boxed{0} & \boxed{0} \quad \boxed{\ldots} \end{array}$$

where the boxed elements stabilize (i.e., do not change when passing to higher-step spectral sequences). This shows that the maps

$$\mathrm{H}^1(S_4, M) \to \mathrm{H}^1(K, M)^{S_3}, \qquad \mathrm{H}^2(S_4, M) \to \mathrm{H}^2(K, M)^{S_3} \tag{5}$$

induced by restriction maps are in fact isomorphisms.

Lemma 2.3. *Let K be the Klein four group, M a vector space over $\mathbf{F}_2$ with trivial action of K. Denote the elements of K by 1, u, v, uv. Then*:

(i) *The map*

$$\mathrm{H}^2(K, M) \ni [\eta] \mapsto \big(\eta(u,u) - \eta(0,0), \eta(v,v) - \eta(0,0), \eta(uv,uv) - \eta(0,0)\big) \in M^3$$

is an isomorphism.

(ii) *Every cohomology class in $\mathrm{H}^2(K, M)$ contains precisely one cocycle η such that $\eta(0,k) = \eta(k,0) = \eta(u,v) = 0$ for every $k \in K$.*

Proof. (i) This is a direct consequence of the 2-cocycle equation. Details are omitted.

(ii) For any cocycle η in a given cohomology class, one checks that the cocycle $\eta' = \eta + \partial f$ with $f : K \to M$ such that $f(0) = \eta(0,0)$, $f(u) = f(v) = 0$, $f(uv) = \eta(u,v)$ satisfies $\eta'(0,0) = \eta'(u,v) = 0$ and the 2-cocycle equation implies that we also have $\eta'(0,k) = \eta'(k,0) = 0$ for any $k \in K$. Uniqueness results from the fact that any coboundary ∂f such that $\partial f(0,0) = \partial f(u,v) = 0$ is identically zero. □

Lemma 2.4. $\dim_k \mathrm{H}^1(S_4, M) = 1$, $\dim_k \mathrm{H}^2(S_4, M) = 2$.

Proof. The first claim is a consequence of Lemma 2.2 and isomorphisms (5). For the second claim, choose $[\eta] \in \mathrm{H}^2(K, M)$ and write $x = \eta(u,u)$, $y = \eta(v,v)$, $z = \eta(uv,uv)$. By Lemma 2.3(i), $[\eta]$ is invariant under the action of S_3 if and only if $sy = x$, $sx = y$, $sz = z$, $rz = x$, $rx = y$, $ry = z$. This is equivalent to $z = r^2x$, $y = rx$, $srx = x$. One easily checks that the space of solutions to these equations is two dimensional. □

3. Computation of the universal deformation ring

We shall consider deformations of the representation $\tau : S_4 \to \mathrm{GL}_2(k)$. Denote by $\mathcal{C}_W$ the category of Artinian local W-algebras with residue field k and let $D : \mathcal{C}_W \to \mathbf{Sets}$ be the deformation functor of τ (for definition of D and other basic concepts used in the proof, consult [2] or [3]).

Theorem 3.1. *The universal deformation ring of τ is $R = W[[t]]/(2t, t^2)$.*

Proof. Let $A \in \mathcal{C}_W$ and choose some a in the maximal ideal of A. Using the description (2) of S_4 in terms of generators and relations one easily checks that the existence of a homomorphism $\bar{\tau}_a : S_4 \to \mathrm{Aut}(V \otimes_k A)$ such that

$$\bar{\tau}_a(u) = \begin{pmatrix} 1+a & a \\ 0 & 1+a \end{pmatrix}, \quad \bar{\tau}_a(v) = \begin{pmatrix} 1+a & 0 \\ a & 1+a \end{pmatrix}, \quad \bar{\tau}_a(r) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \bar{\tau}_a(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6}$$

is equivalent to $2a = a^2 = 0$. Moreover, it is then unique and is a deformation of τ to A. The map $\Phi : \mathrm{Hom}(R, \cdot) \to D$ given by

$$\Phi_A : \mathrm{Hom}(R, A) \ni \varphi \mapsto \bar{\tau}_{\varphi(t)} \in D(A)$$

is thus a well-defined morphism of functors. I claim that each Φ_A is injective. In fact, $\Phi_A(\varphi_1) = \Phi_A(\varphi_2)$ means that $\bar{\tau}_{\varphi_1(t)}$ and $\bar{\tau}_{\varphi_2(t)}$ are conjugate by some matrix $P \in \ker(\mathrm{GL}_2(A) \to \mathrm{GL}_2(k))$. Such a P necessarily commutes with $\bar{\tau}_{\varphi_1(t)}(r) = \bar{\tau}_{\varphi_2(t)}(r) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ and $\bar{\tau}_{\varphi_1(t)}(s) = \bar{\tau}_{\varphi_2(t)}(s) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, hence with the algebra generated by these matrices, which is all of $M_2(A)$. Thus $\varphi_1(t) = \varphi_2(t)$. Since the tangent spaces to both functors are one dimensional (cf. Lemma 2.4), this also proves that Φ_A is bijective for $A = k[x]/x^2$.

We claim now that Φ is smooth. Let $A' \to A$ be a small surjection in $\mathcal{C}_W$ with kernel I and assume that $\bar{\tau}_a \in D(A)$ is in the image of the map $D(A') \to D(A)$, i.e., the obstruction induced by $\bar{\tau}_a$ in $\mathrm{H}^2(S_4, M)$ is zero, and so is its image in $\mathrm{H}^2(K, M)$. We shall only compute the latter one. Choose a lift a' of a to A' and for the sake of calculating the obstruction choose the following lifts $\tilde{\tau}(k)$ of $\bar{\tau}_a(k)$ to A': $\tilde{\tau}(0) = \mathrm{id}$, $\tilde{\tau}(u) = \bar{\tau}_{a'}(u)$, $\tilde{\tau}(v) = \bar{\tau}_{a'}(v)$, $\tilde{\tau}(uv) = \bar{\tau}_{a'}(u)\bar{\tau}_{a'}(v)$. These lifts define a 2-cocycle η in $\mathrm{H}^2(K, M) \otimes I$ given by $\eta(k, l) = \tilde{\tau}(k)\tilde{\tau}(l)\tilde{\tau}(kl)^{-1} - \mathrm{id}$. We have $\eta(0, k) = \eta(k, 0) = \eta(u, v) = 0$ and thus by Lemma 2.3(ii) $\eta = 0$. Hence, in particular,

$$\eta(u, u) = \begin{pmatrix} 2a' + a'^2 & 2a' + 2a'^2 \\ 0 & 2a' + a'^2 \end{pmatrix} = 0$$

and thus $2a' = a'^2 = 0$. Hence $\bar{\tau}_{a'}$ is another lift of $\bar{\tau}_a$ to A'. Since the fibers of the maps $\mathrm{Hom}(R, A') \to \mathrm{Hom}(R, A)$ and $D(A') \to D(A)$ are transitive under the action of respective tangent spaces, one can in fact shift $\bar{\tau}_{a'}$ by an image of an element in $\mathrm{Hom}(R, A')$ so as to obtain any element in the fiber. This proves that the map

$$\mathrm{Hom}(R, A') \to \mathrm{Hom}(R, A) \times_{D(A)} D(A')$$

is surjective and hence Φ is smooth. Thus R is a versal deformation ring of τ. Universality follows from the injectivity of all the maps Φ_A. □

Corollary 3.2. *The universal deformation ring of τ is not a complete intersection ring.*

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References

[1] F.M. Bleher, T. Chinburg, Universal deformation rings need not be complete intersections, C. R. Acad. Sci. Paris, Ser. I 342 (2006) 229–232.

[2] B. Mazur, An introduction to the deformation theory of Galois representations, in: G. Cornell, J.H. Silverman, G. Stevens (Eds.), Modular Forms and Fermat's Last Theorem, Springer-Verlag, 1997, pp. 243–312.

[3] M. Schlessinger, Functors of Artin rings, Trans. Amer. Math. Soc. 130 (1968) 208–222.

[4] J.-P. Serre, Local Fields, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, 1979.