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Rotation fields and the fundamental theorem of Riemannian geometry in $\mathbb{R}^3$

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Abstract

Let Ω be a simply-connected open subset of $\mathbb{R}^3$. We show in this Note that, if a smooth enough field $\mathbf{U}$ of symmetric and positive-definite matrices of order three satisfies the compatibility relation (due to C. Vallée)

$$\mathbf{CURL}\,\boldsymbol{\Lambda} + \mathbf{COF}\,\boldsymbol{\Lambda} = \mathbf{0} \quad \text{in } \Omega,$$

where the matrix field $\boldsymbol{\Lambda}$ is defined in terms of the field $\mathbf{U}$ by

$$\boldsymbol{\Lambda} = \frac{1}{\det \mathbf{U}}\left\{\mathbf{U}(\mathbf{CURL}\,\mathbf{U})^{\mathrm{T}}\mathbf{U} - \frac{1}{2}\left(\mathrm{tr}\left[\mathbf{U}(\mathbf{CURL}\,\mathbf{U})^{\mathrm{T}}\right]\right)\mathbf{U}\right\},$$

then there exists, typically in spaces such as $W^{2,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$ or $\mathcal{C}^2(\Omega;\mathbb{R}^3)$, an immersion $\boldsymbol{\Theta}:\Omega\to\mathbb{R}^3$ such that $\mathbf{U}^2 = \boldsymbol{\nabla}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{\nabla}\boldsymbol{\Theta}$ in Ω. In this approach, one directly seeks the polar factorization $\boldsymbol{\nabla}\boldsymbol{\Theta} = \mathbf{RU}$ of the gradient of the unknown immersion $\boldsymbol{\Theta}$ in terms of a rotation $\mathbf{R}$ and a pure stretch $\mathbf{U}$. ***To cite this article: P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Champs de rotations et le théorème fondamental de la géométrie riemannienne dans $\mathbb{R}^3$. Soit Ω un ouvert simplement connexe de $\mathbb{R}^3$. On montre dans cette Note que, si un champ suffisamment régulier $\mathbf{U}$ de matrices symétriques définies positives d'ordre trois satisfait la relation de compatibilité (due à C. Vallée)

$$\mathbf{CURL}\,\boldsymbol{\Lambda} + \mathbf{COF}\,\boldsymbol{\Lambda} = \mathbf{0} \quad \text{dans } \Omega,$$

où le champ $\boldsymbol{\Lambda}$ de matrices est défini en fonction du champ $\mathbf{U}$ par

$$\boldsymbol{\Lambda} = \frac{1}{\det \mathbf{U}}\left\{\mathbf{U}(\mathbf{CURL}\,\mathbf{U})^{\mathrm{T}}\mathbf{U} - \frac{1}{2}\left(\mathrm{tr}\left[\mathbf{U}(\mathbf{CURL}\,\mathbf{U})^{\mathrm{T}}\right]\right)\mathbf{U}\right\},$$

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alors il existe, typiquement dans des espaces tels que $W^{2,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$ ou $\mathcal{C}^2(\Omega;\mathbb{R}^3)$, une immersion $\boldsymbol{\Theta}:\Omega\to\mathbb{R}^3$ telle que $\mathbf{U}^2=\boldsymbol{\nabla\Theta}^{\mathrm{T}}\boldsymbol{\nabla\Theta}$ in Ω. Dans cette approche, on cherche à identifier directement la factorisation polaire $\boldsymbol{\nabla\Theta}=\mathbf{RU}$ du gradient de l'immersion inconnue $\boldsymbol{\Theta}$ en une rotation $\mathbf{R}$ et une extension pure $\mathbf{U}=\mathbf{C}^{1/2}$. ***Pour citer cet article : P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

En 1992, Vallée [13] a montré que le champ $\mathbf{C}=\boldsymbol{\nabla\Theta}^{\mathrm{T}}\boldsymbol{\nabla\Theta}$ de tenseurs métriques associé à une immersion suffisamment régulière $\boldsymbol{\Theta}:\Omega\to\mathbb{R}^3$ définie sur un ouvert $\Omega\subset\mathbb{R}^3$ vérifie nécessairement la relation de compatibilité

$$\mathbf{CURL}\,\boldsymbol{\Lambda}+\mathbf{COF}\,\boldsymbol{\Lambda}=\mathbf{0}\quad\text{in }\Omega,$$

où le champ $\boldsymbol{\Lambda}$ de matrices est défini en fonction du champ $\mathbf{U}=\mathbf{C}^{1/2}$ par

$$\boldsymbol{\Lambda}=\frac{1}{\det\mathbf{U}}\left\{\mathbf{U}(\mathbf{CURL}\,\mathbf{U})^{\mathrm{T}}\mathbf{U}-\frac{1}{2}\big(\mathrm{tr}\big[\mathbf{U}(\mathbf{CURL}\,\mathbf{U})^{\mathrm{T}}\big]\big)\mathbf{U}\right\}.$$

L'objet principal de cette Note est d'établir la réciproque suivante : Si un champ suffisamment régulier $\mathbf{C}$ de matrices symétriques définies positives d'ordre trois satisfait la relation de compatibilité ci-dessus dans un ouvert $\Omega\subset\mathbb{R}^3$ simplement connexe, alors il existe, typiquement dans des espaces tels que $W^{2,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$ ou $\mathcal{C}^2(\Omega;\mathbb{R}^3)$, une immersion $\boldsymbol{\Theta}:\Omega\to\mathbb{R}^3$ telle que $\mathbf{C}=\boldsymbol{\nabla\Theta}^{\mathrm{T}}\boldsymbol{\nabla\Theta}$ in Ω ; voir Théorèmes 5.1, 5.2, et 5.3.

Ce théorème d'existence global fournit donc une alternative au théorème fondamental de la géométrie riemannienne pour un ouvert $\Omega\subset\mathbb{R}^3$, dans lequel la relation de compatibilité exprime classiquement que le tenseur de courbure de Riemann associé au champ $\mathbf{C}$ s'annule dans Ω.

La démonstration consiste d'abord à déterminer un champ $\mathbf{R}$ de matrices orthogonales dans Ω, puis à déterminer une immersion $\boldsymbol{\Theta}$ telle que $\boldsymbol{\nabla\Theta}=\mathbf{RC}^{1/2}$ dans Ω, en résolvant successivement deux systèmes de Pfaff. En plus de sa nouveauté, cette approche est donc de nature plus « géométrique » que l'approche classique, dans la mesure où elle cherche à identifier directement la factorisation polaire $\boldsymbol{\nabla\Theta}=\mathbf{RU}$ du gradient de l'immersion en une rotation $\mathbf{R}$ et une extension pure $\mathbf{U}=\mathbf{C}^{1/2}$.

A cet égard, il est à signaler que Shield [10] fut le premier à reconnaître l'importance de la factorisation polaire $\boldsymbol{\nabla\Theta}=\mathbf{RU}$ pour l'obtention de relations de compatibilité satisfaites par le champ de matrices $\mathbf{C}=\boldsymbol{\nabla\Theta}^{\mathrm{T}}\boldsymbol{\nabla\Theta}$; dans cette direction, voir aussi Edelen [4].

Il est à noter que l'on établit ici des théorèmes d'existence globale, d'une part, et dans des espaces « à faible régularité », tels que $W^{2,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^3)$, d'autre part. À cette fin, on fait un usage essentiel d'un résultat profond d'existence globale pour des systèmes de Pfaff (rappelé ici dans le Théorème 3.1), récemment établi par S. Mardare [8]. Pour des données régulières, ce résultat est un cas particulier de théorèmes dûs à Cartan [1] et Thomas [12].

Cette approche constitue également un premier pas vers l'analyse mathématique de modèles en élasticité tridimensionnelle non linéaire où le champ des rotations est considéré comme une inconnue à part entière. Voir dans cette direction Pietraszkiewicz et Badur [9], qui ont exploité cette même idée du point de vue de la mécanique des milieux continus.

1. Notations and preliminaries

In Sections 1 to 4, the notation p designates any integer $\geqslant 2$. It is then understood that Latin indices and exponents range in the set $\{1,2,\ldots,p\}$ and that the summation convention with respect to repeated indices and exponents is used in conjunction with this rule. In Section 5, the same rules apply with $p=3$.

All matrices considered here have real elements. The notations $\mathbb{M}^p$, $\mathbb{S}^p$, $\mathbb{S}^p_>$, $\mathbb{A}^p$, and $\mathbb{O}^p$ respectively designate the sets of all square matrices, of all symmetric matrices, of all symmetric and positive-definite symmetric matrices, of all antisymmetric matrices, and of all orthogonal matrices, of order p. The notation $\mathbb{M}^{p\times q}$ designates the set of all matrices with p rows and q columns. The notation (a_{ij}) designates the matrix in $\mathbb{M}^p$ with a_{ij} as its elements, the first index being the row index, and given a matrix $\mathbf{A}=(a_{ij})\in\mathbb{M}^p$, the notation $(\mathbf{A})_{ij}$ designates its element a_{ij}.

The Euclidean norm of $\mathbf{a} \in \mathbb{R}^p$ is denoted $|\mathbf{a}|$ and the Euclidean inner-product of $\mathbf{a} \in \mathbb{R}^p$ and $\mathbf{b} \in \mathbb{R}^p$ is denoted $\mathbf{a} \cdot \mathbf{b}$. The vector product of $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$ is denoted $\mathbf{a} \wedge \mathbf{b}$. The *cofactor matrix* $\mathbf{COF\,A}$ associated with a matrix $\mathbf{A} = (a_{ij}) \in \mathbb{M}^3$ is defined by $(\mathbf{COF\,A})_{ij} = \frac{1}{2}\varepsilon_{mni}\varepsilon_{qrj}a_{mq}a_{nr}$, where (ε_{ijk}) is the orientation tensor.

The coordinates of a point $x \in \mathbb{R}^p$ are denoted x_i. Partial derivative operators, in the usual sense or in the sense of distributions, of the first, second, and third order are denoted $\partial_i := \partial/\partial x_i$, $\partial_{ij} := \partial^2/\partial x_i \partial x_j$, and $\partial_{ijk} := \partial^3/\partial x_i \partial x_j \partial x_k$.

All the vector spaces considered in this paper are over $\mathbb{R}$. The notations $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ respectively designate the space of all functions in $\mathcal{C}^\infty(\Omega)$ whose support is compact and contained in Ω, and the space of distributions over Ω. The notations $\mathcal{C}^l(\Omega)$, $l \geqslant 0$, and $W^{m,\infty}_{\mathrm{loc}}(\Omega)$, $m \geqslant 0$, respectively designate the spaces of continuously differentiable functions over Ω and the usual Sobolev spaces, with $W^{0,\infty}_{\mathrm{loc}}(\Omega) = L^\infty_{\mathrm{loc}}(\Omega)$.

Given a mapping $\boldsymbol{\Theta} = (\Theta_i) \in \mathcal{D}'(\Omega; \mathbb{R}^p)$, the matrix field $\boldsymbol{\nabla}\boldsymbol{\Theta} \in \mathcal{D}'(\Omega; \mathbb{M}^p)$ is defined by $(\boldsymbol{\nabla}\boldsymbol{\Theta})_{ij} = \partial_j \Theta_i$. Given a matrix field $\mathbf{A} = (a_{ij}) \in \mathcal{D}'(\Omega; \mathbb{M}^3)$, the notation $\mathbf{CURL\,A}$ designate the matrix field

$$\mathbf{CURL\,A} := \begin{pmatrix} \partial_2 a_{13} - \partial_3 a_{12} & \partial_3 a_{11} - \partial_1 a_{13} & \partial_1 a_{12} - \partial_2 a_{11} \\ \partial_2 a_{23} - \partial_3 a_{22} & \partial_3 a_{21} - \partial_1 a_{23} & \partial_1 a_{22} - \partial_2 a_{21} \\ \partial_2 a_{33} - \partial_3 a_{32} & \partial_3 a_{31} - \partial_1 a_{33} & \partial_1 a_{32} - \partial_2 a_{31} \end{pmatrix} \in \mathcal{D}'\big(\Omega; \mathbb{M}^3\big).$$

Finally, we recall that a mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbb{R}^p)$ is an *immersion* if the matrix $\boldsymbol{\nabla}\boldsymbol{\Theta}(x) \in \mathbb{M}^p$ is invertible at all points $x \in \Omega$.

Complete proofs of the results announced in this Note are found in [3].

2. Compatibility relations satisfied by the matrix field $\mathbf{U} = (\boldsymbol{\nabla}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{\nabla}\boldsymbol{\Theta})^{1/2}$

Let Ω be an arbitrary open subset of $\mathbb{R}^p$. It is well known that the *metric tensor* field $\mathbf{C} := \boldsymbol{\nabla}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{\nabla}\boldsymbol{\Theta} \in \mathcal{C}^2(\Omega; \mathbb{S}^p_>)$ associated with an immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^p)$ *necessarily* satisfies *compatibility relations* that take the form $R_{qijk} = 0$ in Ω, where the functions $R_{qijk} \in \mathcal{C}^0(\Omega)$ are the covariant components of the associated *Riemann curvature tensor* (for details, see, e.g., [2]).

The next theorem shows that, likewise, the matrix field $\mathbf{U} := \mathbf{C}^{1/2}$ *necessarily* satisfies *ad hoc compatibility relations*. These relations, which were first established in componentwise form by Shield [10] for smooth immersions, are written here in more concise matrix form. In addition, they can be shown to hold in function spaces with little regularity.

Theorem 2.1. *Let Ω be an open subset of $\mathbb{R}^p$ and let there be given an immersion $\boldsymbol{\Theta} \in W^{2,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^p)$. At each point $x \in \Omega$, let $\boldsymbol{\nabla}\boldsymbol{\Theta}(x) = \mathbf{R}(x)\mathbf{U}(x)$, with*

$$\mathbf{U}(x) := \big(\boldsymbol{\nabla}\boldsymbol{\Theta}^{\mathrm{T}}(x)\boldsymbol{\nabla}\boldsymbol{\Theta}(x)\big)^{1/2} \in \mathbb{S}^p_> \quad \textit{and} \quad \mathbf{R}(x) := \boldsymbol{\nabla}\boldsymbol{\Theta}(x)\mathbf{U}(x)^{-1} \in \mathbb{O}^p,$$

denote the unique polar factorization of the matrix $\boldsymbol{\nabla}\boldsymbol{\Theta}(x)$. Then the fields $\mathbf{U}$ and $\mathbf{R}$ defined in this fashion possess the following regularities:

$$\mathbf{U} \in W^{1,\infty}_{\mathrm{loc}}\big(\Omega; \mathbb{S}^p_>\big) \quad \textit{and} \quad \mathbf{R} \in W^{1,\infty}_{\mathrm{loc}}\big(\Omega; \mathbb{O}^p\big).$$

Let the matrix fields $\mathbf{A}_j \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{A}^p)$ be defined in terms of the matrix field $\mathbf{U}$ by

$$\mathbf{A}_j := \frac{1}{2}\big\{\mathbf{U}^{-1}\big(\boldsymbol{\nabla}\mathbf{c}_j - (\boldsymbol{\nabla}\mathbf{c}_j)^{\mathrm{T}}\big)\mathbf{U}^{-1} + \mathbf{U}^{-1}\partial_j\mathbf{U} - (\partial_j\mathbf{U})\mathbf{U}^{-1}\big\},$$

where $\mathbf{c}_j \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^p)$ denotes the j-th column vector field of the matrix field $\mathbf{C} := \mathbf{U}^2 \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{S}^p_>)$. Then the matrix field $\mathbf{U}$ necessarily satisfies the following compatibility relations:

$$\partial_i\mathbf{A}_j - \partial_j\mathbf{A}_i + \mathbf{A}_i\mathbf{A}_j - \mathbf{A}_j\mathbf{A}_i = \mathbf{0} \quad \textit{in } \mathcal{D}'\big(\Omega; \mathbb{A}^p\big).$$

3. A fundamental existence theorem for linear differential systems

Our proof in the next section that the compatibility relations shown in Section 2 to be necessarily satisfied by the matrix field $\mathbf{U} = (\boldsymbol{\nabla}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{\nabla}\boldsymbol{\Theta})^{1/2}$ associated with a given immersion $\boldsymbol{\Theta}$ are also sufficient for the existence of the immersion $\boldsymbol{\Theta}$, relies in an essential way on the following *fundamental existence theorem for linear differential systems with little regularity*, which is due to S. Mardare [8, Theorem 3.6] (for smooth data, this theorem is a special case of earlier existence results of Cartan [1] and Thomas [12]).

Theorem 3.1. *Let Ω be a connected and simply-connected subset of $\mathbb{R}^p$ and let $q \geqslant 1$ be an integer. Let there be given matrix fields $\mathbf{A}_j \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^q)$, $\mathbf{B}_j \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^p)$, and $\mathbf{C}_j \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^{p\times q})$ that satisfy*:

$$\begin{aligned}
&\partial_i \mathbf{A}_j + \mathbf{A}_i\mathbf{A}_j = \partial_j \mathbf{A}_i + \mathbf{A}_j\mathbf{A}_i \quad \text{in } \mathcal{D}'\big(\Omega; \mathbb{M}^q\big),\\
&\partial_i \mathbf{B}_j + \mathbf{B}_j\mathbf{B}_i = \partial_j \mathbf{B}_i + \mathbf{B}_i\mathbf{B}_j \quad \text{in } \mathcal{D}'\big(\Omega; \mathbb{M}^p\big),\\
&\partial_i \mathbf{C}_j + \mathbf{C}_i\mathbf{A}_j + \mathbf{B}_j\mathbf{C}_i = \partial_j \mathbf{C}_i + \mathbf{C}_j\mathbf{A}_i + \mathbf{B}_i\mathbf{C}_j \quad \text{in } \mathcal{D}'\big(\Omega; \mathbb{M}^{p\times q}\big),
\end{aligned}$$

and let a point $x_0 \in \Omega$ and a matrix $\mathbf{F}_0 \in \mathbb{M}^{p\times q}$ be given. Then there exists one and only one matrix field $\mathbf{Y} \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{M}^{p\times q})$ that satisfies:

$$\begin{aligned}
&\partial_j \mathbf{Y} = \mathbf{Y}\mathbf{A}_j + \mathbf{B}_j\mathbf{Y} + \mathbf{C}_j \quad \text{in } \mathcal{D}'\big(\Omega; \mathbb{M}^{p\times q}\big),\\
&\mathbf{Y}(x_0) = \mathbf{F}_0.
\end{aligned}$$

S. Mardare has also shown how this existence result can be extended to the space $W^{1,\infty}(\Omega; \mathbb{M}^{p\times q})$ when, in addition, the geodesic diameter of Ω is finite.

Note that Theorem 3.1 contains two important special cases, viz., a *generalized Poincaré lemma*, corresponding to $\mathbf{A}_i = \mathbf{0}$ and $\mathbf{B}_i = \mathbf{0}$, and a *general existence theorem for Pfaff systems*, corresponding to $\mathbf{B}_i = \mathbf{0}$ and $\mathbf{C}_i = \mathbf{0}$.

4. Sufficiency of the compatibility relations

Under the assumption that the open set Ω is simply-connected, we now show that, if a symmetric and positive-definite matrix field $\mathbf{U}$ defined on Ω satisfies the compatibility relations that were found to be necessary in Theorem 2.1, then conversely there exists an immersion $\boldsymbol{\Theta} : \Omega \to \mathbb{R}^p$ such that $\mathbf{U} = (\nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta})^{1/2}$. Note that this existence result holds for fields $\mathbf{U}$ with little regularity.

Theorem 4.1. *Let Ω be a connected and simply-connected subset of $\mathbb{R}^p$ and let there be given a matrix field $\mathbf{U} \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{S}^p_>)$ that satisfies*

$$\partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i + \mathbf{A}_i\mathbf{A}_j - \mathbf{A}_j\mathbf{A}_i = \mathbf{0} \quad \text{in } \mathcal{D}'\big(\Omega; \mathbb{A}^p\big),$$

where the matrix fields $\mathbf{A}_j \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{A}^p)$ are defined in terms of the matrix field $\mathbf{U}$ by

$$\mathbf{A}_j := \frac{1}{2}\big\{\mathbf{U}^{-1}\big(\nabla\mathbf{c}_j - (\nabla\mathbf{c}_j)^{\mathrm{T}}\big)\mathbf{U}^{-1} + \mathbf{U}^{-1}\partial_j\mathbf{U} - (\partial_j\mathbf{U})\mathbf{U}^{-1}\big\},$$

the field $\mathbf{c}_j \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^p)$ denoting the j-th column vector field of the matrix field $\mathbf{U}^2 \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{S}^p_>)$.

Let there be given a point $x_0 \in \Omega$, a vector $\mathbf{a}_0 \in \mathbb{R}^p$, and a matrix $\mathbf{F}_0 \in \mathbb{M}^p$ that satisfies $(\mathbf{F}_0^{\mathrm{T}}\mathbf{F}_0)^{1/2} = \mathbf{U}(x_0)$. Then there exists one and only one immersion $\boldsymbol{\Theta} \in W^{2,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^p)$ that satisfies

$$\begin{aligned}
&\mathbf{U} = \big(\nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta}\big)^{1/2} \quad \text{in } W^{1,\infty}_{\mathrm{loc}}\big(\Omega; \mathbb{S}^p_>\big),\\
&\boldsymbol{\Theta}(x_0) = \mathbf{a}_0 \quad \text{and} \quad \nabla\boldsymbol{\Theta}(x_0) = \mathbf{F}_0.
\end{aligned}$$

Sketch of proof. The proof comprises three main parts, summarized below. The existence and uniqueness of the solution of the Pfaff systems considered in parts (i) and (iii) rely in an essential way on Theorem 3.1. Part (ii) is used in part (iii) to verify that the compatibility relations of Theorem 3.1 are indeed satisfied by the Pfaff system considered in (iii).

(i) Let there be given a point $x_0 \in \Omega$ and a matrix $\mathbf{R}_0 \in \mathbb{O}^p$. Then there exists one and only one matrix field $\mathbf{R} \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{O}^p)$ that satisfies $\partial_j\mathbf{R} = \mathbf{R}\mathbf{A}_j$ in $L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^p)$ and $\mathbf{R}(x_0) = \mathbf{R}_0$.

(ii) The matrix fields

$$\mathbf{A}_j := \frac{1}{2}\{\mathbf{U}^{-1}(\nabla\mathbf{c}_j - (\nabla\mathbf{c}_j)^{\mathrm{T}})\mathbf{U}^{-1} + \mathbf{U}^{-1}\partial_j\mathbf{U} - (\partial_j\mathbf{U})\mathbf{U}^{-1}\}$$

may be also written as

$$\mathbf{A}_j = \mathbf{U}\boldsymbol{\Gamma}_j\mathbf{U}^{-1} - (\partial_j\mathbf{U})\mathbf{U}^{-1},$$

where the matrix fields $\boldsymbol{\Gamma}_j \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{M}^p)$ are defined by

$$\boldsymbol{\Gamma}_j := \frac{1}{2}\mathbf{U}^{-2}(\partial_j(\mathbf{U}^2) + \nabla\mathbf{c}_j - (\nabla\mathbf{c}_j)^{\mathrm{T}}).$$

(iii) The matrix field $\mathbf{R} \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{O}^p)$ being that determined in (i), there exists one and only one vector field $\boldsymbol{\Theta} \in W^{2,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^p)$ that satisfies $\nabla\boldsymbol{\Theta} = \mathbf{RU}$ in $W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{M}^p)$ and $\boldsymbol{\Theta}(x_0) = \mathbf{a}_0$. The "initial" condition $\nabla\boldsymbol{\Theta}(x_0) = \mathbf{F}_0$ is then satisfied by letting $\mathbf{R}_0 := \mathbf{F}_0(\mathbf{F}_0^{\mathrm{T}}\mathbf{F}_0)^{-1/2}$ in part (i). □

Naturally, if no "initial" conditions such as $\boldsymbol{\Theta}(x_0) = \mathbf{a}_0$ and $\nabla\boldsymbol{\Theta}(x_0) = \mathbf{F}_0$ are imposed, the immersions $\boldsymbol{\Theta} \in W^{2,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^p)$ are then *uniquely defined up to isometries of* $\mathbb{R}^p$ *only*, according to the well-known *rigidity theorem* (which holds for immersions in the space $\mathcal{C}^1(\Omega; \mathbb{R}^p)$; cf., e.g., [2, Theorem 1.7-1]).

5. Special case of an open subset of $\mathbb{R}^3$

In this section, the dimension p of the underlying space is equal to three. In this case, the sufficient compatibility relations of Theorem 4.1 can be re-written in a remarkably simple and concise form, in terms of the matrix operators **CURL** and **COF** applied to an *ad hoc* matrix field $\boldsymbol{\Lambda}$, itself a function of the given matrix field $\mathbf{U}$. These relations are due to Vallée [13], who showed that they are *necessarily* satisfied by the matrix field $\mathbf{U} = (\nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta})^{1/2}$ associated with a *given* immersion $\boldsymbol{\Theta}$. The main result of this Note is that they are also *sufficient*, according to the following *global existence theorem*.

Theorem 5.1. *Let Ω be a connected and simply-connected subset of $\mathbb{R}^3$ and let there be given a matrix field $\mathbf{U} \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{S}^3_>)$ that satisfies*

$$\mathbf{CURL}\,\boldsymbol{\Lambda} + \mathbf{COF}\,\boldsymbol{\Lambda} = \mathbf{0} \quad \text{in } \mathcal{D}'(\Omega; \mathbb{M}^3),$$

where the matrix field $\boldsymbol{\Lambda} \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^3)$ is defined in terms of the field $\mathbf{U}$ by

$$\boldsymbol{\Lambda} = \frac{1}{\det\mathbf{U}}\left\{\mathbf{U}(\mathbf{CURL}\,\mathbf{U})^{\mathrm{T}}\mathbf{U} - \frac{1}{2}(\mathrm{tr}[\mathbf{U}(\mathbf{CURL}\,\mathbf{U})^{\mathrm{T}}])\mathbf{U}\right\}.$$

Let there be given a point $x_0 \in \Omega$, a vector $\mathbf{a}_0 \in \mathbb{R}^3$, and a matrix $\mathbf{F}_0 \in \mathbb{M}^3$ that satisfies $(\mathbf{F}_0^{\mathrm{T}}\mathbf{F}_0)^{1/2} = \mathbf{U}(x_0)$. Then there exists one and only one immersion $\boldsymbol{\Theta} \in W^{2,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^3)$ that satisfies

$$\mathbf{U} = (\nabla\boldsymbol{\Theta}^{\mathrm{T}}\nabla\boldsymbol{\Theta})^{1/2} \quad \text{in } W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{S}^3_>),$$
$$\boldsymbol{\Theta}(x_0) = \mathbf{a}_0 \quad \text{and} \quad \nabla\boldsymbol{\Theta}(x_0) = \mathbf{F}_0.$$

Sketch of proof. The proof comprises four main parts, summarized below. In what follows, the notation $[\mathbf{A}]_j : \Omega \to \mathbb{R}^3$ designates the j-th column vector field of a given matrix field $\mathbf{A}$.

(i) Given matrix fields $\mathbf{A}_j \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{A}^3)$, let $a_{1j} := (\mathbf{A}_j)_{32}, a_{2j} := (\mathbf{A}_j)_{13}$, and $a_{3j} := (\mathbf{A}_j)_{21}$, and define the matrix field $\boldsymbol{\Lambda} \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^3)$ by $\boldsymbol{\Lambda} := (a_{ij})$. Equivalently, $[\boldsymbol{\Lambda}]_j \wedge \mathbf{v} = \mathbf{A}_j\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$.

Then we first show, by means of a direct computation, that the relations assumed in Theorem 4.1 on the matrix fields $\mathbf{A}_j$ are equivalent to the relation assumed in Theorem 5.1 on the matrix field $\boldsymbol{\Lambda}$.

(ii) Given a matrix field $\mathbf{C} \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{S}^3_>)$, define the matrix fields $\mathbf{A}_j$ as in Theorem 4.1, i.e., by

$$\mathbf{A}_j := (\mathbf{U}\boldsymbol{\Gamma}_j - \partial_j\mathbf{U})\mathbf{U}^{-1} \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{A}^3),$$

where

$$\mathbf{U} := \mathbf{C}^{1/2} \quad \text{and} \quad \boldsymbol{\Gamma}_j := \frac{1}{2}\mathbf{C}^{-1}\big(\partial_j \mathbf{C} + \nabla \mathbf{c}_j - (\nabla \mathbf{c}_j)^{\mathrm{T}}\big) \in L^\infty_{\mathrm{loc}}\big(\Omega; \mathbb{M}^3\big) \quad \text{and} \quad \mathbf{c}_j := [\mathbf{C}]_j.$$

We then show, by means of somewhat delicate computations, that the matrix field $\boldsymbol{\Lambda}$ is given in terms of the matrix field $\mathbf{U}$ by the expression announced in the statement of Theorem 5.1.

(iii) Conversely, given a matrix field $\mathbf{U} \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{S}^3_>)$, define the matrix field $\boldsymbol{\Lambda} \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^3)$ by

$$\boldsymbol{\Lambda} = \frac{1}{\det \mathbf{U}}\left\{\mathbf{U}(\mathbf{CURL}\,\mathbf{U})^{\mathrm{T}}\mathbf{U} - \frac{1}{2}\big(\mathrm{tr}\big[\mathbf{U}(\mathbf{CURL}\,\mathbf{U})^{\mathrm{T}}\big]\big)\mathbf{U}\right\},$$

and let the matrix fields $\mathbf{A}_j \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{A}^3)$ be defined by the relations $\mathbf{A}_j \mathbf{v} = \mathbf{a}_j \wedge \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$ where $\mathbf{a}_j := [\boldsymbol{\Lambda}]_j$. Our third objective consists in showing that the matrix fields $\mathbf{A}_j$ are given by

$$\mathbf{A}_j = (\mathbf{U}\boldsymbol{\Gamma}_j - \partial_j \mathbf{U})\mathbf{U}^{-1},$$

where

$$\boldsymbol{\Gamma}_j = \frac{1}{2}\mathbf{U}^{-2}\big(\partial_j\big(\mathbf{U}^2\big) + \nabla \mathbf{c}_j - (\nabla \mathbf{c}_j)^{\mathrm{T}}\big) \quad \text{and} \quad \mathbf{c}_j := \big[\mathbf{U}^2\big]_j.$$

We claim that this conclusion can be reached without any further computation, by means of the following simple argument. First, any equivalence class in $L^\infty_{\mathrm{loc}}(\Omega)$ can be identified with a well defined function from Ω into $\mathbb{R}$. So, given a matrix field $\boldsymbol{\Lambda} \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^3)$, let the matrix fields $\mathbf{A}_j \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{A}^3)$ be defined as above. Now, at each point $x \in \Omega$, the linear mapping

$$\big(\mathbf{A}_1(x), \mathbf{A}_2(x), \mathbf{A}_3(x)\big) \in \big(\mathbb{A}^3\big)^3 \to \boldsymbol{\Lambda}(x) \in \mathbb{M}^3$$

defined by the relations $[\boldsymbol{\Lambda}(x)]_j \wedge \mathbf{v} = \mathbf{A}_j(x)\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$ is clearly one-to-one and onto between two finite-dimensional linear spaces of dimension nine. This observation shows that, given any field $\boldsymbol{\Lambda} \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^3)$, there is one and only one field $(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) \in (L^\infty_{\mathrm{loc}}(\Omega; \mathbb{A}^3))^3$ that satisfies $\mathbf{A}_j \mathbf{v} = \mathbf{a}_j \wedge \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$. The assertion thus follows from the computations made in part (ii).

(iv) The existence and uniqueness of the immersion $\boldsymbol{\Theta} \in W^{2,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^3)$ follow by combining the equivalences established in part (i) and in part (iv) with Theorem 4.1. □

As expected, part (iii) of the above proof can be also established by means of a direct, although somewhat delicate, computation.

The existence result of Theorem 5.1 can be extended to the spaces $\mathcal{C}^{m+1}(\Omega; \mathbb{R}^3)$, $m \geqslant 1$, and $W^{2,\infty}(\Omega; \mathbb{R}^3)$, as follows.

Theorem 5.2. *Assume in Theorem* 5.1 *that the matrix field* $\mathbf{U}$ *belongs to the set* $\mathcal{C}^m(\Omega; \mathbb{S}^3_>)$ *for some integer* $m \geqslant 1$, *all the other assumptions and definitions of Theorem* 5.1 *holding* verbatim. *Then the immersion* $\boldsymbol{\Theta}$ *found in Theorem* 5.1 *belongs to the space* $\mathcal{C}^{m+1}(\Omega; \mathbb{R}^3)$.

Theorem 5.3. *Assume in Theorem* 5.1 *that the geodesic diameter of* Ω *is finite and that the matrix field* $\boldsymbol{U}$ *belongs to the set* $W^{1,\infty}(\Omega; \mathbb{S}^3_>)$, *all the other assumptions and definitions of Theorem* 5.1 *holding* verbatim. *Then the immersion* $\boldsymbol{\Theta}$ *found in Theorem* 5.1 *belongs to the space* $W^{2,\infty}(\Omega; \mathbb{R}^3)$.

To conclude our analysis, we mention that, as expected, the compatibility relations found in Theorem 5.1 are equivalent to the vanishing of the Riemann curvature tensor. To this end, we resort in an essential way to Theorem 5.1 (otherwise, a proof by direct computation turns out to be surprisingly lengthy and delicate; see [14]).

Theorem 5.4. *Let* Ω *be an open subset of* $\mathbb{R}^3$. *Then a matrix field* $\mathbf{U} \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{S}^3_>)$ *satisfies*

$$\mathbf{CURL}\,\boldsymbol{\Lambda} + \mathbf{COF}\,\boldsymbol{\Lambda} = \mathbf{0} \quad \text{in } \mathcal{D}'\big(\Omega; \mathbb{M}^3\big),$$

where the matrix field $\boldsymbol{\Lambda} \in L^{\infty}_{\mathrm{loc}}(\Omega, \mathbb{M}^3)$ *is defined in terms of the field* $\mathbf{U}$ *by*

$$\boldsymbol{\Lambda} = \frac{1}{\det \mathbf{U}}\left\{\mathbf{U}(\mathbf{CURL}\,\mathbf{U})^{\mathrm{T}}\mathbf{U} - \frac{1}{2}\big(\mathrm{tr}\big[\mathbf{U}(\mathbf{CURL}\,\mathbf{U})^{\mathrm{T}}\big]\big)\mathbf{U}\right\},$$

if and only if the matrix field $\mathbf{C} = (g_{ij}) := \mathbf{U}^2 \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{S}^3_>)$ *satisfies*

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp} = 0 \quad \textit{in } \mathcal{D}'(\Omega),$$

where

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \in L^{\infty}_{\mathrm{loc}}(\Omega) \quad \textit{and} \quad \Gamma^p_{ij} := g^{pq}\Gamma_{ijq} \in L^{\infty}_{\mathrm{loc}}(\Omega),$$

where $(g^{pq}) := (g_{ij})^{-1} \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{S}^3_>)$.

Note that the cancellation of both the curvature and the torsion, expressed in the classical approach by means of the relations $R_{qijk} = 0$ and $\Gamma^p_{ij} = \Gamma^p_{ji}$ in Ω, likewise manifest themselves in the present approach, *albeit* in a more subtle way; in this respect, see Hamdouni [7].

As advocated notably by Fraeijs de Veubeke [5], Pietraszkiewicz and Badur [9], or Simo and Marsden [11], *rotation fields* can be introduced as *bona fide* unknowns in nonlinear three-dimensional elasticity. This introduction typically involves the replacement of the deformation gradient $\boldsymbol{\nabla\Theta}$ in the stored energy function by a *rotation field* and a *pure stretch field* $\mathbf{U}$, "the constraint" $\boldsymbol{\nabla\Theta} = \mathbf{RU}$ being enforced by means of an appropriate *Lagrange multiplier*, thus producing a *multi-field variational principle*.

The existence theory for models based on such principles appears to be an essentially virgin territory (with the noticeable exception of Grandmont, Maday and Métier [6], who considered a time-dependent elasticity problem in dimension two where a "global rotation" is one of the unknowns). It is thus hoped that the present work constitutes a first step towards the mathematical analysis of such models.

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