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Partial Differential Equations

Solutions with interior bubble and boundary layer for an elliptic Neumann problem with critical nonlinearity

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Abstract

We study positive solutions of the equation $\epsilon^2\Delta u - u + u^{\frac{n+2}{n-2}} = 0$, where $n = 3, 4, 5$ and $\epsilon > 0$ is small, with Neumann boundary condition in a unit ball B. We prove the existence of solutions with an interior bubble at the center and a boundary layer at the boundary ∂B. ***To cite this article: J. Wei, S. Yan, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Solution explosant au centre et le long du bord pour un problème elliptique de Neumann avec non-linéarité critique. Nous considérons le problème $\epsilon^2\Delta u - u + u^{\frac{n+2}{n-2}} = 0$, $u > 0$, dans la boule unité B de $\mathbb{R}^n$ où $n = 3, 4, 5$, $\epsilon > 0$ est petit et u vérifie les conditions au bord de Neumann. Nous montrons l'existence d'une solution radiale se concentrant au centre et le long de la frontière de B quand ϵ tend vers 0. ***Pour citer cet article : J. Wei, S. Yan, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

Soient $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, $p > 1$ et $\epsilon > 0$ petit. On considère le problème

$$\epsilon^2\Delta u - u + u^p = 0,\ u > 0 \quad \text{dans } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{sur } \partial\Omega. \tag{1}$$

Si $p < \frac{n+2}{n-2}$, ou sait qu'il existe des solutions se concentrant en des points situés à l'intérieur ou sur la frontière du domaine quand ϵ tend vers 0 (voir [6,7]). Si $p = \frac{n+2}{n-2}$, l'existence de solutions explosant en un point [12] ou plusieurs points [5,8] du bord est counue. La possibilité d'explosions à l'intérieur du domaine demeure ouverte, même si l'existence de solutions se concentrant exclusivement loin du bord est exclue (voir [3,13]). Par ailleurs, Malchiodi et Montenegreto [9] ont établi l'existence de solutions qui explosent sur tout le bord, pour tout $p > 1$ (au moins pour une suite ϵ_k tendant vers 0).

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Dans cette Note, nous considérons le cas particulier avec exposant critique,

$$\epsilon^2\Delta u - u + n(n-2)u^{\frac{n+2}{n-2}} = 0,\ u > 0 \quad \text{dans } B, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{sur } \partial B \tag{2}$$

où B est la boule unité de $\mathbb{R}^n$ avec $n = 3, 4, 5$. Nous montrons l'existence d'une solution radiale u_ϵ qui explose à la fois au centre de B et le long de sa frontière quand ϵ tend vers 0. Plus précisément, soit w_ϵ la famille de solutions radiales de (2) qui explosent le long du bord [9]. Asymptotiquement w_ϵ se comporte comme $w(\frac{1-|y|}{\epsilon})$, où w l'unique solution de l'équation différentielle :

$$w'' - w + n(n-2)w^{\frac{n+2}{n-2}} = 0,\ w > 0 \quad \text{dans } \mathbb{R}, \qquad w(0) = \max_{t\in\mathbb{R}^1} w(t),\ w(t) \to 0,\ \text{quand } |t| \to +\infty.$$

Ou sait d'autre part que les solutions de $\Delta u + n(n-2)u^{(n+2)/(n-2)} = 0$, $u > 0$ dans $\mathbb{R}^n$ s'écrivent sous la forme $U_{a,\lambda}(x) = (\frac{\lambda}{1+\lambda^2|x-a|^2})^{\frac{n-2}{2}}$, $\lambda > 0$, $a \in \mathbb{R}^n$. Notre résultat principal s'énonce ainsi :

Théorème. *Soit $n = 3$, 4 ou 5. Pour $\epsilon > 0$ assez petit, le problème (2) admet une solution radiale telle que*

$$u_\epsilon = \epsilon^{(n-2)/2}U_{0,\lambda_\epsilon} + w_\epsilon + o(1), \quad \text{avec } \lambda_\epsilon = e^{(2+o(1))/\epsilon} \text{ si } n = 3 \text{ et } \lambda_\epsilon = e^{(6-n+o(1))/2\epsilon} \text{ si } n = 4, 5.$$

1. Introduction and statement of main result

In recent years, there have been many works devoted to the study of the following singularly perturbed Neumann problem:

$$\epsilon^2\Delta u - u + u^p = 0,\ u > 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{3}$$

where $\Omega \subset \mathbb{R}^n$, $p > 1$ and $\epsilon > 0$ is small. Problem (3) arises in the study of many reaction–diffusion systems in chemistry or biology, see [11] and the references therein for backgrounds and progress up to 2004.

When $p < \frac{n+2}{n-2}$, it is known that there are many solutions with point condensations in the interior or on the boundary: for example, Gui and Wei [6] proved that given any two positive integers l_1, l_2, there are solutions to (3) with l_1 interior spikes and l_2 boundary spikes. Lin, Ni and Wei [7] showed that there are at least $\frac{C}{\epsilon^n(|\ln\epsilon|)^n}$ number of interior spike solutions. When $p = \frac{n+2}{n-2}$, it is known that nonconstant solutions exist for ϵ small enough [1], and the least energy solution blows up, as $\epsilon \to 0$, at a point which maximizes the mean curvature of the boundary [2]. Higher energy solutions have also been exhibited, blowing up at one [12] or several (separated) boundary points [5,8]. However, the question of existence of *interior blow-up* solutions is still open. Under some assumptions, it is proved in [3] and [13] that there are no interior bubble solutions. In another direction, Malchiodi and Montenegro [9] proved that there exists solutions concentrating on the whole boundary (at least along a subsequence $\epsilon_k \to 0$). This boundary layer solution exists for *any* $p > 1$.

In this Note, we show that in the critical case $p = \frac{n+2}{n-2}$, one can build up an interior bubble solution on the top of the boundary layer solutions, at least when the domain is the unit ball and the dimension $n = 3$, 4 or 5. Namely, we consider the following

$$\epsilon^2\Delta u - u + n(n-2)u^{\frac{n+2}{n-2}} = 0,\ u > 0 \quad \text{in } B, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B \tag{4}$$

where B is the unit ball in $\mathbb{R}^n$ centered at the origin and $n = 3, 4, 5$. Note that interior bubble has zero dimension concentration set while the boundary layer has $n-1$ dimensional concentration set. Our solution constructed in this paper has both point condensation and $(n-1)$-dimensional concentration. This type of solution is new.

To state our results, we need to introduce two functions. First, it is known that problem (4) has radial symmetric solution concentrating at $r = 1$. (This is actually true for *general domain*. See [9].) This boundary layer solution is denoted as w_ϵ. Asymptotically, $w_\epsilon \approx w(\frac{1-|y|}{\epsilon})$, where w is the unique solution satisfying

$$w'' - w + n(n-2)w^{\frac{n+2}{n-2}} = 0,\ w > 0 \quad \text{in } \mathbb{R}^1, \qquad w(0) = \max_{t\in\mathbb{R}^1} w(t),\ w(t) \to 0,\ \text{as } |t| \to +\infty. \tag{5}$$

On the other hand, it is well-known that the functions $U_{a,\lambda}(x) = (\frac{\lambda}{1+\lambda^2|x-a|^2})^{\frac{n-2}{2}}$, $\lambda > 0$, $a \in \mathbb{R}^n$ are the only solutions to the problem $\Delta u + n(n-2)u^{(n+2)/(n-2)} = 0$, $u > 0$ in $\mathbb{R}^n$.

The main result in this Note is:

Theorem 1. *Let $n=3$, 4 or 5. There exists an $\epsilon_0>0$ such that for $\epsilon\in(0,\epsilon_0)$, problem* (4) *has a radially symmetric solution of the form*

$$u_\epsilon=\epsilon^{(n-2)/2}U_{0,\lambda_\epsilon}+w_\epsilon+\mathrm{o}(1),\quad \text{with } \lambda_\epsilon=\mathrm{e}^{(2+\mathrm{o}(1))/\epsilon} \text{ when } n=3 \text{ and } \lambda_\epsilon=\mathrm{e}^{(2+\mathrm{o}(1))/((6-n)\epsilon)} \text{ when } n=4,5.$$

Remarks. 1. From the calculations in this paper, we can see that (4) does not have a solution which just has an interior bubble at the origin. Our main result shows that it is the boundary layer that creates a solution with a bubble at the origin. When $n\geqslant 6$, our computations suggest that such kind of solutions don't exist.

2. In [10], it is proved that problem (4) has radial solutions concentrating on arbitrarily many spheres near $r=1$. We can also show the existence of an interior bubble solution on the top of clustered boundary interface solutions. See [16].

3. We believe that Theorem 1 also holds in general domains. We conjecture that one can add one (maybe many) interior bubbles to the boundary layer solution constructed in [9], in the lower dimension case $n=3,4,5$.

The present Note concerns partial results obtained in [16], where the procedure and the proof of more general theorems is carried out in full detail.

2. Error estimates and energy computations for approximate solutions

We first analyze the boundary layer solution w_ϵ. We have

Lemma 2. *As $\epsilon\to 0$, $-\epsilon\log w_\epsilon(|x|)\to 1-|x|$ uniformly for $|x|\leqslant\frac{1}{2}$. Furthermore, the linearized operator $L_{\epsilon,1}(\phi):=\epsilon^2\Delta\phi-\phi+n(n+2)w_\epsilon^{\frac{4}{n-2}}\phi$ is an invertible operator from $H^2_{r,\nu}(B)=H^2(B)\cap\{u=u(r),\ \frac{\partial u}{\partial\nu}=0 \text{ on } B\}$ to $L^2(B)$.*

Let λ be such that $\lambda\in\Lambda:=(\mathrm{e}^{\frac{a_n-\delta}{\epsilon}},\mathrm{e}^{\frac{a_n+\delta}{\epsilon}})$ where $a_3=2$, $a_n=\frac{2}{6-n}$ for $n=4$ or 5, and $0<\delta<\frac{1}{100}$ is a fixed small number. Then by the following scaling $u(x)=(\lambda\epsilon)^{\frac{n-2}{2}}\hat{u}(y)$, $x=\frac{1}{\lambda}y$, problem (4) becomes

$$S_{\lambda,\epsilon}[u]:=\Delta u-(\lambda\epsilon)^{-2}u+n(n-2)u_+^{\frac{n+2}{n-2}}=0\quad\text{in } B_\lambda,\qquad \frac{\partial u}{\partial\nu}=0\quad\text{on }\partial B_\lambda \tag{6}$$

where $B_\lambda=\frac{B}{\lambda}$. Solutions to (6) can be found as critical points of the following energy functional

$$J_\epsilon(u)=\frac{1}{2}\int_{B_\lambda}\left(|\nabla u|^2+(\lambda\epsilon)^{-2}u^2\right)-\frac{(n-2)^2}{2}\int_{B_\lambda}u_+^{\frac{2n}{n-2}}. \tag{7}$$

We consider a linear Neumann problem whose solution can be viewed as a projection of $U_{a,\lambda}$ onto $H^2_{r,\nu}(B)$:

$$\Delta V_{a,\lambda}-\epsilon^{-2}V_{a,\lambda}+n(n-2)U_{a,\lambda}^{\frac{n+2}{n-2}}=0\quad\text{in } B,\qquad \frac{\partial V_{a,\lambda}}{\partial\nu}=0\quad\text{on }\partial B. \tag{8}$$

Define $W_1:=\lambda^{-\frac{n-2}{2}}V_{0,\lambda}(\frac{y}{\lambda})$, $W_2:=(\lambda\epsilon)^{-\frac{n-2}{2}}w_\epsilon(\frac{y}{\lambda})$, $W:=W_1+W_2$.

We first need to analyze the projection $V_{0,\lambda}(x)$. For $n=3$, let $V_{0,\lambda}(x)=U_{0,\lambda}(x)-\frac{1}{\lambda^{1/2}}\frac{1}{|x|}(1-\mathrm{e}^{-|x|/\epsilon})+\varphi_{\lambda,\epsilon}(x)$. Then it is easy to see that (see e.g. estimate (2.10) of [14]) $\varphi_{\lambda,\epsilon}(x)=\mathrm{O}(\frac{1}{\epsilon^2\lambda^{3/2}(1+\lambda|x|)}+\frac{\mathrm{e}^{-1/\epsilon}}{\epsilon\lambda^{1/2}})$, which gives

$$W_1(y)=U_{0,1}(y)-\frac{1}{|y|}\left(1-\mathrm{e}^{-\frac{|y|}{\lambda\epsilon}}\right)+\mathrm{O}\left(\frac{1}{\epsilon^2\lambda^2(1+|y|)}+\frac{\mathrm{e}^{-\frac{1}{\epsilon}}}{\epsilon\lambda}\right),\quad\text{when } n=3. \tag{9}$$

Observe that when $n=3$, $|y|>\delta\lambda$ or $|x|>\delta$, (9) gives

$$\left|W_1(y)\right|=\mathrm{O}\left(\frac{1}{\lambda^2}+\frac{\mathrm{e}^{-\frac{|y|}{\lambda\epsilon}}}{\epsilon\lambda}\right),\qquad V_{0,\lambda}(x)=\mathrm{O}\left(\frac{1}{\lambda^{3/2}}+\frac{Y\,\mathrm{e}^{-\frac{|x|}{\epsilon}}}{\epsilon\lambda^{\frac{1}{2}}}\right). \tag{10}$$

When $n=4$ or 5, we have (similar to estimates in Lemma A.1 of [15])

$$W_1(y)=U_{0,1}(y)-\varphi_2\left(\frac{y}{\lambda}\right)+\mathrm{O}\left(\frac{1}{\lambda^{n-2}}\right) \tag{11}$$

where $\varphi_2(x)$ satisfies $\Delta\varphi_2-\epsilon^{-2}\varphi_2+\epsilon^{-2}U_{0,\lambda}=0$ in B, $\frac{\partial\varphi_2}{\partial\nu}=0$ on ∂B. Hence $|\varphi^2(x)|\leqslant\frac{C}{\lambda^2\epsilon^3(1+\lambda|x|)^{n-4}}$.

Next we define two Sobolev norms. (See [4] and [15].) Let $\|\phi\|_*:=\sup_{y\in B_\lambda}(1+|y|)^{\frac{n-2+\delta}{2}}|\phi(y)|$, $\|f\|_{**}:=\sup_{y\in B_\lambda}((1+|y|)^{\frac{n+2+\delta}{2}}|f(y)|)$. Then we compute that

$$\left|S_{\lambda,\epsilon}[W]\right|=n(n-2)\left|(W_1+W_2)_+^{\frac{n+2}{n-2}}-W_2^{\frac{n+2}{n-2}}-U_{0,1}^{\frac{n+2}{n-2}}\right|\leqslant CW_1^{\frac{4}{n-2}}W_2+CW_2^{\frac{4}{n-2}}W_1+CU_{0,1}^{\frac{4}{n-2}}|W_1-U_{0,1}|. \tag{12}$$

The most difficult term in (12) is $W_2^{\frac{4}{n-2}}W_1$, which we now estimate: when $n=3$ and $|y|<\delta\lambda$, we have

$$W_2^{\frac{4}{n-2}}W_1\leqslant C\lambda^{-2}W_1w_\epsilon^4\leqslant C\lambda^{-2}\big(1+|y|\big)^{-1}\mathrm{e}^{-\frac{3}{\epsilon}}\leqslant C\lambda^{-3/2}\big(1+|y|\big)^{-3}$$

while when $n=3$ and $|y|\geqslant\delta\lambda$, we use (10) to obtain $W_2^{\frac{4}{n-2}}W_1\leqslant C\lambda^{-2}W_1w_\epsilon^4\leqslant C\lambda^{-2}\lambda^{-3/2}\leqslant C\lambda^{-1}(1+|y|)^{-5/2}$. Thus when $n=3$ we have $\|W_2^{\frac{4}{n-2}}W_1\|_{**}<\lambda^{-(1-\delta)}$. Similarly, using (9) and (11), a straightforward computation shows that

$$\left\|S_{\lambda,\epsilon}[W]\right\|_{**}\leqslant C\lambda^{-\frac{\beta_n+2\delta}{2}}\quad\text{where } \beta_n=1 \text{ when } n=3,\text{ and } \beta_n=2 \text{ when } n\geqslant4. \tag{13}$$

Next, we compute the energy expansion-$J_\epsilon[W]$. Observe that

$$J_\epsilon[W_1]=-\frac{(n-2)^2}{2}\int_{B_\lambda}(U_{0,1}+W_1-U_{0,1})^{\frac{2n}{n-2}}+\frac{n(n-2)}{2}\int_{B_\lambda}U_{0,1}^{\frac{n+2}{n-2}}(U_{0,1}+W_1-U_{0,1}).$$

Using (9) and (11), we obtain

$$J_\epsilon(W_1)=(n-2)\int_{\mathbb{R}^n}U_{0,1}^{2n/(n-2)}+\begin{cases}\dfrac{3}{2}\displaystyle\int_{\mathbb{R}^3}U_{0,1}^5\frac{1}{\lambda\epsilon}+\mathrm{O}\big(\mathrm{e}^{-(3-\delta)/\epsilon}\big), & \text{when } n=3,\\ \dfrac{1}{2}\epsilon^{-2}\displaystyle\int_B U_{0,\lambda}^2+\mathrm{O}\big(\epsilon^{-1}\lambda^{-(n-2)}\big), & \text{when } n\geqslant4.\end{cases} \tag{14}$$

Using (14), we have the following asymptotic behavior of the energy expansion $J_\epsilon(W)$:

$$\begin{aligned}J_\epsilon(W)-J_\epsilon(W_2)&=(n-2)\int_{\mathbb{R}^n}U_{0,1}^{2n/(n-2)}\\&+\begin{cases}\big(2\pi+\mathrm{o}(1)\big)\epsilon^{-1}\lambda^{-1}-\big(\overline{B}_3+\mathrm{o}(1)\big)\epsilon^{-1/2}\mathrm{e}^{-1/\epsilon}\lambda^{-1/2}+\mathrm{O}\big(\mathrm{e}^{-(2+4\delta)/\epsilon}\big), & \text{when } n=3,\\ \dfrac{\epsilon^{-2}}{2}\displaystyle\int_B U_{0,\lambda}^2-\big(\overline{B}_n+\mathrm{o}(1)\big)\epsilon^{(2-n)/2}\mathrm{e}^{-1/\epsilon}\lambda^{-(n-2)/2}+\mathrm{O}\big(\epsilon^{-1}\lambda^{-2}\big), & \text{when } n\geqslant4,\end{cases}\end{aligned} \tag{15}$$

where $\overline{B}_n>0$ is a positive constant.

Proof of (15). We first prove (15) when $n=3$. Since w_ϵ is a solution to (4), we have $J_\epsilon(W_1+W_2)=J_\epsilon(W_1)+J_\epsilon(W_2)-I_\epsilon$, where

$$I_\epsilon:=\frac{1}{2}\int_{B_\lambda}\big((W_1+W_2)^6-W_1^6-W_2^6-6W_2^5W_1-6W_2W_1^5\big)+3\int_{B_\lambda}W_2W_1^5=:I_{\epsilon,1}+I_{\epsilon,2}$$

and

$$I_{\epsilon,2}=3\epsilon^{-1/2}\int_{|x|\leqslant\delta}w_\epsilon V_{0,\lambda}^5+\mathrm{O}\big(\epsilon^{1/2}\lambda^{-5/2}\big)=\big(\overline{B}_3+\mathrm{o}(1)\big)\epsilon^{-1/2}\mathrm{e}^{-1/\epsilon}\lambda^{-1/2}+\mathrm{O}\big(\epsilon^{1/2}\lambda^{-5/2}\big). \tag{16}$$

On the other hand, by (10), $V_{0,\lambda} = O(\frac{1}{\lambda^{3/2}} + \frac{e^{-1/\epsilon}}{\epsilon\lambda^{1/2}})$ for $|x| > \delta$. Thus

$$I_{\epsilon,1} = \epsilon^{-2}\frac{15}{2}\int_B w_\epsilon^4 V_{0,\lambda}^2 + O\Bigl(\epsilon^{-3/2}\int_B w_\epsilon^3 V_{0,\lambda}^3 + \epsilon^{-1}\int_B w_\epsilon^2 V_{0,\lambda}\Bigr) = O\bigl(\epsilon^{-1}\lambda^{-3} + e^{-2/\epsilon}\lambda^{-1}\bigr). \tag{17}$$

Substituting (17) and (16) into I_ϵ and using the fact that $\int_{\mathbb{R}^3} U_{0,1}^5 = \frac{4\pi}{3}$, we obtain (15) for $n = 3$.
When $n = 4$, the proof of (15) is similar and easier by noting that for $n \geqslant 4$, we have

$$\int_B U_{0,\lambda}^2 \sim \frac{\ln\lambda}{\lambda^2}, \quad \text{if } n = 4; \qquad \int_B U_{0,\lambda}^2 \sim \frac{1}{\lambda^2}, \quad \text{if } n \geqslant 5, \tag{18}$$

and hence $\int_{B_\lambda} W_2^{\frac{4}{n-2}} W_1^2 = O(\epsilon^{-1}\lambda^{-(n-2)})$ is a higher order term. □

3. Finite-dimensional reduction

In this section, we perform a finite-dimensional reduction procedure similar to that of [4] and [15].

We first consider the following linear problem: Let $Z = U_{0,1} + \frac{n-2}{2} y\nabla U_{1,0}$. Given $h = h(r)$, find a pair (ϕ, c) satisfying

$$\Delta\phi - (\lambda\epsilon)^{-2}\phi + n(n+2)W_+^{\frac{4}{n-2}}\phi = h + cZ, \quad \text{in } B_\lambda, \quad \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \partial B_\lambda, \quad \int_{B_\lambda} \phi Z = 0. \tag{19}$$

We have the following a priori estimates.

Lemma 3. *Let (ϕ, c) satisfy* (19). *Then for ϵ sufficiently small, there holds $\|\phi\|_* \leqslant C\|h\|_{**}$.*

Proof. Observe first that the Green function of $\Delta G - (\lambda\epsilon)^{-2}G + \delta_y = 0$ in B_λ, $\frac{\partial G}{\partial\nu} = 0$ on ∂B_λ has the following decay property: $G(x, y) \leqslant \frac{C}{|x-y|^{n-2}}$. Secondly, the operator $\Delta - (\lambda\epsilon)^{-2} + n(n+2)W_2^{\frac{4}{n-2}}$ is uniformly invertible by Lemma 2. The rest of the proof is similar to Proposition 3.1 of [15]. We omit the details. □

By estimate (13), Lemma 3 and a contraction mapping principle, we derive the following reduction lemma:

Lemma 4. *There exists $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$, the following problem*

$$\Delta(W+\phi) - (\lambda\epsilon)^{-2}(W+\phi) + n(n-2)(W+\phi)_+^{\frac{n+2}{n-2}} = c_\lambda Z \quad \text{in } B_\lambda, \quad \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \partial B_\lambda, \quad \int_{B_\lambda} \phi Z = 0 \tag{20}$$

has a unique solution $(\phi_\lambda, c_\lambda)$. Moreover the map $\lambda \to \phi_\lambda$ is C^1 and $\|\phi_\lambda\|_* \leqslant C\lambda^{-\frac{\beta_n+\delta}{2}}$.

Now we define $M(\lambda) = J_\epsilon(W + \phi_\lambda) - J_\epsilon(W_2) - (n-2)\int_{\mathbb{R}^n} U_{0,1}^{2n/(n-2)}$, where $J_\epsilon(W_2)$ is independent of λ. Then we also have

Lemma 5. *If $\lambda = \lambda_\epsilon$ is a critical point of $M(\lambda)$ in Λ, then $u_\epsilon = W + \phi_{\lambda_\epsilon}$ is a solution to* (6).

Thus we are reduced to finding a critical point of $M(\lambda)$.

4. Proof of Theorem 1

We first expand $M(\lambda)$: using (13) and Lemma 4, we deduce that

$$M(\lambda) = J_\epsilon(W) + \int_{B_\lambda} S_\lambda[W]\phi_\lambda + O\bigl(\|\phi_\lambda\|_*^2\bigr) - J_\epsilon(W_2) - (n-2)\int_{\mathbb{R}^n} U_{0,1}^{2n/(n-2)}.$$

When $n=3$, we use (15) to derive that $M(\lambda)=(2\pi+\mathrm{o}(1))\epsilon^{-1}\lambda^{-1}-(\overline{B}_3+\mathrm{o}(1))\epsilon^{-1/2}\,\mathrm{e}^{-1/\epsilon}\lambda^{-1/2}+\mathrm{O}(\mathrm{e}^{-3/\epsilon})$. Observe that the function $(2\pi+\mathrm{o}(1))\epsilon^{2}\lambda^{-1}-(\overline{B}_3+\mathrm{o}(1))\epsilon^{-1/2}\,\mathrm{e}^{-1/\epsilon}\lambda^{-1/2}$ attains its minimum at $\lambda_\epsilon=\mathrm{e}^{\frac{2+\mathrm{o}(1)}{\epsilon}}\in\Lambda$. On the other hand, when $n=4$ or 5, we have

$$M(\lambda)=\frac{1}{2}\epsilon^{-2}\int\limits_B U_{0,\lambda}^2-\big(\overline{B}_n+\mathrm{o}(1)\big)\epsilon^{(2-n)/2}\,\mathrm{e}^{-1/\epsilon}\lambda^{-(n-2)/2}+\mathrm{O}\big(\epsilon^{-1}\lambda^{-2}\big). \tag{21}$$

Using (18), we find that the function $\frac{1}{2}\epsilon^{-2}\int_B U_{0,\lambda}^2-\overline{B}_n\epsilon^{(2-n)/2}\,\mathrm{e}^{-1/\epsilon}\lambda^{-(n-2)/2}$ has a critical point $\bar{\lambda}_\epsilon=\mathrm{e}^{\frac{2+\mathrm{o}(1)}{(6-n)\epsilon}}$. Thus, the reduced energy functional $M(\lambda)$ also has a critical point $\lambda_\epsilon=\mathrm{e}^{\frac{2+\mathrm{o}(1)}{(6-n)\epsilon}}$. By Lemma 5, $W_1+W_2+\phi_{\lambda_\epsilon}$ is a solution to (6). Let $u_\epsilon(x)=W_1(\lambda_\epsilon x)+W_2(\lambda_\epsilon x)+\phi_{\lambda_\epsilon}(\lambda_\epsilon x)$. Then u_ϵ satisfies all the properties of Theorem 1. □

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