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Differential Geometry/Dynamical Systems

On lightlike geometry: isometric actions, and rigidity aspects

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Abstract

Degenerate Riemannian metrics exist naturally in various contexts. Unfortunately, their study stops to the 'admission of failure' that they are too poor, for instance, to generate a coherent intrinsic or extrinsic differential geometry, e.g. a kind of Levi-Civita connection. In this first text, we start the investigation of rigidity aspects of these structures, from the point of view of isometric actions of 'big' (e.g. semi-simple) Lie groups. ***To cite this article: E. Bekkara et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Sur la géométrie de lumière : actions isométriques et rigidité. Les métriques riemanniennes dégénérées apparaissent naturellement dans divers contextes. Malheureusement leur étude est souvent limitée par le triste constat qu'elles sont trop pauvres pour donner lieu aux outils classiques de géométrie différentielle, extrinsèque ou intrinsèque, comme par exemple un analogue de la connexion de Levi-Civita. Dans ce papier, nous abordons quelques aspects de la rigidité de ces structures, du point de vue des actions isométriques des groupes de Lie semi-simples. ***Pour citer cet article : E. Bekkara et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

Un phénomène bien connu en géométrie pseudo-riemannienne est l'existence de sous-variétés sur lesquelles la métrique ambiante induit une métrique dégénérée. Dans la théorie de la relativité générale, l'apparition de telles sous-variétés est même naturelle puisque les horizons des trous noirs, ou les bords des domaines de dépendance sont systématiquement des hypersurfaces (souvent singulières) sur lesquelles la métrique lorentzienne ambiante dégénère (voir [5] par exemple). Notons aussi que de telles hypersurfaces interviennent dans la compactification d'espaces comme l'espace de Minkowski (ou les espaces de Minkowski généralisés, c'est à dire les espaces plats simplement connexes de signature (p, q)).

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Pour ce type de sous-variétés, toute la théorie standard (opérateur de Weingarten, équations de Gauss–Codazzi, etc. ...), valable pour les sous-variétés non dégénérées, s'écroule (voir toutefois les travaux de [7,12]). Notre but dans cette note est de montrer que par certains aspects, la situation n'est pas aussi critique qu'elle ne le semble à première vue, puisque l'on peut dégager certains phénomènes de rigidité pour ces structures dégénérées.

Pour se placer dans un cadre général, oublions la motivation initiale qui est l'étude des sous-variétés des variétés pseudo-riemanniennes, et définissons directement ce qu'est une variété pseudo-riemannienne dégénérée. Il s'agira pour nous d'une variété lisse M, munie d'un tenseur symétrique g, de type $(2,0)$, dont la signature est *constante* de la forme (n_-, n_0, n_+), où n_- (resp. n_+) est le nombre de $-$ (resp. de $+$) dans la signature, et n_0 est la dimension du noyau. Nous nous intéressons ici au cas le plus proche du contexte riemannien, c'est-à-dire $n_- = 0$, et $n_0 = 1$. Pour des raisons qui viennent de la physique, de telles métriques « riemanniennes dégénérées avec noyau unidimensionnel » seront appelées *métriques de lumière*.

Commençons par décrire deux paradigmes de métriques de lumière :

• *Flots riemanniens*.
Soit (M, g) une variété munie d'une métrique de lumière g, et supposons que chaque point de M possède sur un voisinage des coordonées locales $(x_0, x_1, \ldots, x_n)$ où la métrique g s'exprime par $\Sigma_{i,j>0} g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j$, avec les fonctions g_{ij} indépendantes de x_0. Le feuilletage $\mathcal{N}$ de dimension 1 défini par le noyau de g est alors ce que l'on appelle un feuilletage transversalement riemannien. Si X est un champ de vecteurs sur M, tangent en tout point à $\mathcal{N}$, la dérivée de Lie de g dans la direction de X s'annule : $\mathcal{L}_X g = 0$. C'est la situation la moins rigide qui soit, puisque « en poussant » le long du feuilletage $\mathcal{N}$, on hérite d'un gros groupe d'isométries pour g (en particulier ce groupe n'est pas un groupe de Lie de dimension finie).

• *Le cône isotrope dans l'espace de Minkowski*.
L'espace de Minkowski est l'espace $\mathbb{R}^{1+n}$ muni de la forme quadratique $q(x) = -x_0^2 + x_1^2 + \cdots + x_n^2$. le cône positif Co^n est l'ensemble des x qui annulent q et pour lesquels $x_0 > 0$. Le cône Co^n admet une métrique de lumière naturelle : la restriction de la métrique q, et cette métrique est bien sûr préservée par l'action du groupe $O^+(1,n)$ (le sous-groupe d'indice 2 de $O(1,n)$ qui préserve le futur du cône de lumière). Cet exemple s'oppose radicalement au précédent, puisque nous montrons qu'il est rigide globalement en dimension $n \geqslant 3$, et même localement, dès que $n \geqslant 4$. Il existe en effet un *Théorème de Liouville* (voir le fait 1) qui stipule que dès que $n \geqslant 4$, toute isométrie locale de Co^n est la restriction d'un élément de $O^+(1,n)$.

L'étude de ces deux exemples laisse supposer que la bonne hypothèse « générique » à imposer à une métrique de lumière g pour obtenir de la rigidité est $\mathcal{L}_N g \neq 0$, pour tout champ de vecteurs N tangent au feuilletage unidimensionnel $\mathcal{N}$ défini par g. Nous reportons l'étude systématique de la rigidité à une autre occasion, pour nous consacrer dans cette note à une forme de rigidité qui concerne les actions isométriques. Cela répond à une question naturelle lorsque l'on fait l'étude d'une structure géométrique : exhiber, et si possible classifier, les objets les plus symmétriques, par exemple, les structures homogènes. Notre résultat principal est le Théorème 3.1 que nous énonçons ici informellement : à facteur riemannien homogène près, le cône Co^n est le seul exemple de variété munie d'une métrique de lumière, sur laquelle un groupe de Lie semi-simple agit non proprement et transitivement. L'énoncé précis du Théorème 3.1 donne également des informations dans le cas où l'on relaxe l'hypothèse de transitivité.

1. Introduction

Pseudo-Riemannian (also said semi-Riemannian) manifolds have non-degenerate, but not necessarily positive definite metrics. However, a submanifold in them is not necessarily pseudo-Riemannian, that is the induced metric may be degenerate. This is a delicate situation where all submanifold theory (shape operator, Gauss and Codazzi equations, ...) seems to fail. It is our purpose here to notice that this situation is not as bad as it appears, that is, the inherited 'slack' geometry is in fact generically 'rigid'. In fact, with respect to our approach here the submanifold structure is irrelevant, the pertinent framework is therefore that of (abstract) manifolds with 'degenerate pseudo-Riemannian' structures. More precisely, a degenerate pseudo-Riemannian metric, or alternatively a pseudo-Riemannian semi-metric on a manifold M is just a smooth symmetric tensor g of degree 2 with *constant* signature (n_-, n_0, n_+), so that we have on each tangent space a quadratic form and an orthogonal basis for it, with n_- (resp. n_+) negative (resp. positive) elements, and n_0 isotropic ones. The latter space of dimension n_0 forms the Kernel (or normal, radical, characteristic) bundle of g.

We are interested here in the less degenerate case, where $n_- = 0$ and $n_0 = 1$, i.e. a *degenerate Riemannian metric with radical of dimension 1* which we will call a **lightlike** metric (not only because this is a pretty word, but also because this is the name of degenerate submanifolds in Lorentz manifolds, and there, the terminology has in particular a physical and optical origin, see for instance [14]). So a lightlike metric has its characteristic bundle N of dimension one, and thus determines a 1-dimensional foliation $\mathcal{N}$.

2. Two Preliminary examples

2.1. The most flexible example: transversally Riemannian flows

The linear situation reduces to the case of $\mathbb{R}^{1+n}$ with coordinates $(x_0, x_1, \ldots, x_n)$, endowed with the lightlike quadratic form $x_1^2 + \cdots + x_n^2$. A non-linear generalization of this is given by transversally Riemannian flows. In coordinates, this corresponds to a lightlike metric $g = \Sigma_{i,j>0} g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j$, where the g_{ij} do not depend of x_0. More synthetically, a transversally Riemannian flow on a manifold M consists of a 1-dimensional foliation $\mathcal{N}$ which is the characteristic foliation of a lightlike metric, and such that the flow of any vector field N tangent to $\mathcal{N}$ preserves g, that is, the Lie derivative satisfies $\mathcal{L}_N g = 0$. In fact, it is sufficient that some N satisfies this equation in order that any collinear vector field satisfies it as well. Such a lightlike metric is somehow tame, since, at least locally, the quotient space $M/\mathcal{N}$ (here to be precise one must replace M by a small open set where the foliation $\mathcal{N}$ is trivial), inherits a Riemannian metric, in other words the projection of g is well defined, and has no Kernel since we have exactly killed it by taking the quotient. In contrast, the isometry group of g contains at least all flows tangent to $\mathcal{N}$ which form an infinitely dimensional group, surely not so beautiful. Anyway this mixture of dynamics is well understood, at least in the case where M is compact, due to many works on transversally Riemannian foliations: [4], [13], ...

Observe that a lightlike metric g generates a semi-distance d_g, enjoying all properties of standard distances except the 'separability' one: $d_g(x, y) = 0$ does not imply $x = y$.

2.2. A rigid example: the Minkowski lightlike cone

Let $Min_{1,n}$ be the Minkowski space of dimension $1+n$, that is $\mathbb{R}^{1+n}$ endowed with the form $q = -x_0^2 + x_1^2 + \cdots x_n^2$. The isotropic (positive) cone Co^n is the set $\{q(x) = 0,\ x_0 > 0\}$. The metric induced by q on Co^n is lightlike. The group $O^+(1, n)$ (subgroup of $O(1, n)$ preserving the positive cone) acts isometrically on Co^n. This action is in fact transitive so that $Co^n = O^+(1, n)/Euc_{n-1}$ becomes a lightlike homogeneous space, with isotropy group $Euc_{n-1} = O(n-1) \ltimes \mathbb{R}^{n-1}$, the group of rigid motions of the Euclidean space of dimension $n - 1$.

A key observation is:

Fact 1 *(Liouville Theorem for lightlike geometry). For, $n \geqslant 3$, any isometry of Co^n belongs to $O^+(1, n)$. In fact, this is true even locally for $n \geqslant 4$: any isometry between two connected open subsets of Co^n is the restriction of an element of $O^+(1, n)$.*

- *For $n = 3$, the group of local isometries is in one-to-one correspondence with the group of local conformal transformations of $\mathbf{S}^2$.*
- *For $n = 2$, there is no rigidity at all, even globally, since to any diffeomorphism of the circle corresponds an isometry of Co^2.*

Sketch of proof. The metric on Co^n is just the metric $0 \oplus \mathrm{e}^{2t} g_{\mathbf{S}^{n-1}}$ on $\mathbb{R} \times \mathbf{S}^{n-1}$. An isometry f of Co^n is of the form $(t, x) \mapsto (\lambda(t, x), \phi(x))$. A simple calculation proves that f is isometric iff:

$$\phi^* g_{\mathbf{S}^{n-1}} = \mathrm{e}^{2(t-\lambda(t,x))} g_{\mathbf{S}^{n-1}}.$$

So, any local isometry of Co^n is of the form $(t, x) \mapsto (t - \mu(x), \phi(x))$, with ϕ a local conformal transformation of the sphere satisfying $\phi^* g_{\mathbf{S}^{n-1}} = \mathrm{e}^{2\mu} g_{\mathbf{S}^{n-1}}$. Thus, the different rigidity phenomena are just consequences of classical analogous rigidity results for conformal transformations on the sphere. □

3. Isometric actions of Lie groups

Let (M, g) be a lightlike manifold and G a group acting isometrically on it. In order to avoid trivial situations, we assume G to be a connected Lie group, and also that it does not reduce to just a flow (or say all flows generated by multiples of a vector field) as in the transversally Riemannian case. Let us go further and assume G to be a semi-simple Lie group. Then we have this kind of converse of the previous fact:

Theorem 3.1. *[3] Let G be a semi-simple Lie group with finite center acting faithfully, isometrically and* ***non-properly*** *on a lightlike manifold* (M, g).

- *If the action is transitive, i.e. M is a homogeneous lightlike space* G/H*, then, up to a finite cover, it is isometric to a product of a cone* Co^n ($n = 1$ *is allowed) by a homogeneous Riemannian manifold.*
- *In the case of a general action, assume G has no factor locally isomorphic to* $SL(2,\mathbb{R})$. *Then*:
  - *G has a factor* G' *locally isomorphic to some* $O(1, n)$*, which acts non-properly. In fact* G' *has an orbit isometric to* Co^n *(up to a finite cover).*
  - *The product of all factors not locally isomorphic to* $O(1, n)$*, acts properly on M.*

3.1. Outline of the proof

The detailed proof together with related facts will appear in [3]. We present here briefly some steps. Recall firstly that a non-proper action means the existence of two sequences (x_k) and (y_k) of M, with respective limits x_∞ and y_∞ in M, and a sequence (g_k) tending to infinity in G (i.e. escaping every compact subset of G), such that $y_k = g_k.x_k$.

A first remark, already done in [2], is that a non-proper and isometric action (with respect to a lightlike metric) of a semi-simple Lie group on a manifold, leads (when there is no local $SL(2,\mathbb{R})$ factor) to non-compact stabilizers. So, in spirit (and with a little bit work), the theorem reduces to the homogeneous, and non-proper case, on which we now focus. We will also assume that no local factor of G is isomorphic to $SL(2,\mathbb{R})$. The idea is to describe as precisely as possible the Lie algebra $\mathfrak{g}_x$ of the stabilizer of a point $x \in M$. Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g} = \Sigma_{\alpha\in\Pi^+}\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \Sigma_{\alpha\in\Pi^+}\mathfrak{g}_\alpha$ the corresponding decomposition into rootspaces.

• *Step 1*. The intersection of $\mathfrak{g}_x$ with any Cartan subalgebra reduces to $\{0\}$.
Indeed, an $\mathbb{R}$-semi-simple element in $\mathfrak{g}_x$ has strictly contracted, and strictly expanded eigenspaces, providing a full family of rootspaces which, when evaluated at x, are isotropic. This gives factors of G which act non-trivially on the real line, and finally a local $SL(2,\mathbb{R})$ factor; a contradiction.

• *Step 2*. Exactly one root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{g}_x$.

Using techniques of Kowalsky [11], improved in [2], the non-properness assumption yields a family of rootspaces, inducing vector fields isotropic at x. The main idea here is that thanks to the Cartan decomposition $G = KAK$, the non-properness of the action of G implies the non-properness of the action of $A = Exp_G(\mathfrak{a})$ (the image by the exponential map of the Cartan subalgebra $\mathfrak{a}$). The non-properness assumption thus yields some $x \in M$, and some $H \in \mathfrak{a}$, such that any vector $Y \in \mathfrak{g}_\alpha$, with $\alpha(H) > 0$, has to be isotropic at x (see [2], Theorem 1.8 for more details). A further improvement of this result yields a full rootspace contained in $\mathfrak{g}_x$. Now, $\mathfrak{g}_x$ cannot contain more than one rootspace. This is the consequence of several remarks. The first one is that if $Y \in \mathfrak{g}_\alpha \cap \mathfrak{g}_x$, then $\theta.Y \in \mathfrak{g}_{-\alpha}$ (θ denotes the Cartan involution) can't be in $\mathfrak{g}_x$ without contradicting *Step 1*. Also, if $Y \in \mathfrak{g}_\alpha \cap \mathfrak{g}_x$ and $Z \in \mathfrak{g}_\beta \cap \mathfrak{g}_x$, the dual vectors of α and β in $\mathfrak{a}$, namely T_α and T_β, must be isotropic at x. If α and β are linearly independent, this yields an immediate contradiction with *Step 1*. Also if α and β are proportional but not equal, we can exhibit a contradiction. So, $\mathfrak{g}_x$ contains exactly one rootspace. As a further consequence of this analysis, we get a splitting $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_1 = \mathfrak{g}_{-\alpha} \oplus \mathfrak{a}' \oplus \mathfrak{m}' \oplus \mathfrak{g}_\alpha$ is isomorphic to $\mathfrak{o}(1, n)$, and $\mathfrak{g}_x \cap \mathfrak{g}_1$ is just $\mathfrak{m}' \oplus \mathfrak{g}_\alpha$, which is isomorphic to the Lie algebra of Euclidean motions.

• *Step 3*. Using the previous steps and the understanding of linear unipotent elements preserving a lightlike bilinear form, it is an 'algebraic work', quite similar to that done in Section 3 of [6], to prove that $\mathfrak{g}_x = (\mathfrak{g}_x \cap \mathfrak{g}_1) \oplus \mathfrak{m}''$, where $\mathfrak{m}''$ is a compact Lie algebra in $\mathfrak{g}_2$.

4. Further comments

First of all, let us note that there exist at least the references [7] and [12], which are devoted to adaptation of differential geometry techniques to the lightlike case. However, neither the homogeneous situation, nor the rigidity side of this geometry were systematically investigated in the literature. Actually, germs of rigidity can be extracted for instance from [1]. In a next future, we would like to show that 'generic' lightlike metrics are 'rigid geometric structures', say, in the sense of Gromov's Theory [8], but also in a more classical sense [10]. Roughly speaking genericity is a condition allowing one to avoid the transversally Riemannian case, as explained above. A strong condition ensuring genericity is that the Lie derivative $\mathcal{L}_N g$ along the characteristic direction, is nowhere vanishing. The situation is reminiscent of that of CR-structures.

A nice counterpart of rigidity of geometric structures is to have finitely dimensional Lie groups (or pseudo-groups) of isometries, as in the case of the cone above (Fact 1). This property is sometimes valid only because of global reasons, as in the case of the group of holomorphic transformations of compact complex manifolds. In the future, we intend to use the principal result presented in this note as a starting point to get a full classification of compact lightlike homogeneous spaces.

As for differential lightlike geometry, we notice that, for instance in [9], an isometric immersions theory is developed, in its existence part, i.e. the analogous of Nash's Theorem in the Riemannian case. As we already said, a lightlike metric is in particular that of a degenerate submanifold in a Lorentz space. For them, rigidity theory would allow one to develop an extrinsic geometry, comparably complicated as the existing conformal and projective extrinsic geometries. Horizons of black holes and more generally horizons of domains of dependence in Lorentz manifolds are lightlike, but are rarely sufficiently smooth to be handled by our approach. It is however interesting to understand the very special case where these horizons are indeed smooth [5].

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