

Mathematical Analysis/Functional Analysis

Generalized differentiation and bi-Lipschitz nonembedding in L^1 [☆]

Jeff Cheeger ^a, Bruce Kleiner ^b

^a Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, USA

^b Department of Mathematics, University of Michigan, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109-1043, USA

Received 2 June 2006; accepted 7 June 2006

Available online 10 August 2006

Presented by Jean-Michel Bismut

Abstract

We consider Lipschitz mappings, $f : X \rightarrow V$, where X is a doubling metric measure space which satisfies a Poincaré inequality, and V is a Banach space. We show that earlier differentiability and bi-Lipschitz nonembedding results for maps, $f : X \rightarrow \mathbb{R}^N$, remain valid when \mathbb{R}^N is replaced by any separable dual space. We exhibit spaces which bi-Lipschitz embed in L^1 , but not in any separable dual V . For certain domains, including the Heisenberg group with its Carnot–Carathéodory metric, we establish a new notion of differentiability for maps into L^1 . This implies that the Heisenberg group does not bi-Lipschitz embed in L^1 , thereby proving a conjecture of J. Lee and A. Naor. When combined with their work, this has implications for theoretical computer science.

To cite this article: J. Cheeger, B. Kleiner, *C. R. Acad. Sci. Paris, Ser. I 343 (2006)*.

© 2006 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Résumé

Différentiation généralisée et impossibilité d'un plongement bilipschitzien dans L^1 . Nous considérons des applications lipchitziennes, $f : X \rightarrow V$, où X est un espace métrique mesuré tel que l'on contrôle le volume des boules par doublement du rayon et qui satisfait à une inégalité de Poincaré, et où V est un espace de Banach. On montre que des résultats antérieurs de différentiabilité et de non plongement bilipschitzien pour des applications $f : X \rightarrow \mathbb{R}^N$, restent valables quand on suppose que V est un dual séparable. Nous donnons des exemples d'espaces plongés de manière bilipschitzienne dans L^1 , mais qui ne sont plongeables dans aucun dual séparable. Pour certains espaces, dont le groupe d'Heisenberg muni de la métrique de Carnot–Carathéodory, on établit une nouvelle notion de différentiabilité pour des applications dans L^1 . Ceci implique que le groupe de Heisenberg ne possède aucun plongement bilipschitzien dans L^1 , un résultat conjecturé par J. Lee et A. Naor. Quand il est combiné avec des résultats de ces deux auteurs, notre travail a des applications en informatique théorique. **Pour citer cet article :** J. Cheeger, B. Kleiner, *C. R. Acad. Sci. Paris, Ser. I 343 (2006)*.

© 2006 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Version française abrégée

Le théorème de différentiabilité de Rademacher dit qu'une fonction lipchitzienne définie sur \mathbb{R}^k à valeurs dans \mathbb{R} est dérivable presque partout. Dans cette Note, nous annonçons des résultats liés au théorème de Rademacher. Nous

[☆] The first author was partially supported by NSF Grant DMS 0105128 and the second by NSF Grant DMS 0505610.
E-mail addresses: cheeger@cims.nyu.edu (J. Cheeger), bkleiner@umich.edu (B. Kleiner).

obtenons en particulier des résultats de non plongement bilipchitzien. Nous décrivons brièvement nos résultats dans ce qui suit.

Soit (X, d^X, μ) un espace métrique mesuré qui vérifie une condition de doublement du volume ainsi qu'une inégalité de Poincaré au sens de [11]. Un tel espace est dit de type *PI*.

On dit qu'un espace de Banach V a la propriété de Radon–Nikodym si l'énoncé du théorème de Rademacher est vrai pour toute fonction lipchitzienne de \mathbb{R}^k dans V . Les espaces duaux qui sont séparables et les espaces réflexifs ont la propriété de Radon–Nikodym.

Nous montrons que si X est un espace de type *PI*, il existe un recouvrement dénombrable de X par des cartes mesurables Lipchitziennes qui envoient localement X dans un espace \mathbb{R}^k , tel que si V est un espace dual séparable, toute application lipchitzienne $f : V \rightarrow X$ possède une propriété de type Rademacher relativement à ces cartes. On peut ainsi définir un espace cotangent mesurable T^*X , qui est un fibré vectoriel mesurable sur X . Le fibré tangent TX est son dual.

Pour tout $x \in X$, on peut définir un cône tangent X_x en prenant la limite au sens de Gromov–Hausdorff d'une suite d'espaces pointés. Presque partout, la dimension de Hausdorff de chaque X_x est supérieure ou égale à la dimension de $T_x X$. Si l'ensemble G des points de X où il y a inégalité stricte est de mesure non nulle, nous montrons qu'il n'y a pas de plongement bilipchitzien de X dans un dual séparable. On indique également qu'un groupe de Carnot ne peut pas être plongé de manière bilipchitzienne dans un espace possédant la propriété de Radon–Nikodym.

L'espace L^1 ne possède pas la propriété de Radon–Nikodym. Nous annonçons l'existence d'espaces *PI*, dont en particulier les espaces de Laakso, auxquels le critère de non plongement dans un dual séparable peut être appliqué, mais qui se plongent de manière bilipchitzienne dans L^1 .

Enfin nous indiquons que le groupe de Heisenberg, \mathbb{H} , ne possède pas de plongement bilipchitzien dans L^1 , un résultat conjecturé par J. Lee et A. Naor. Ce résultat possède des applications en informatique.

1. Introduction

Rademacher's differentiation theorem states that a real valued Lipschitz function on \mathbb{R}^k is differentiable almost everywhere. The literature contains a vast number of variants of this theorem, in which either the domain, the target, or the class of mappings is generalized. In this paper we announce several results in a similar vein, as well as examples showing their sharpness and applications to bi-Lipschitz nonembedding theorems.¹

2. PI spaces

Henceforth (X, d^X, μ) will denote a metric measure space satisfying a doubling condition and a Poincaré inequality in the sense of [11] (see below for the definition), and we will refer to such spaces as *PI spaces*.

Examples of PI spaces include Gromov–Hausdorff limits of sequences of Riemannian manifolds with a uniform lower Ricci curvature bound, Bourdon–Pajot spaces, Laakso spaces and Carnot groups such as the Heisenberg group \mathbb{H} ; see [5, 3, 13, 10]. For our present purposes, the latter two examples are particularly significant; see Theorems 5.1, 6.1.

We now recall some relevant definitions concerning PI spaces. Let $B_r(x)$ denote the metric ball, $\{x' \mid d^X(x, x') < r\}$. The *doubling condition* is: for all $x \in X$ and $r \leq R$,

$$\mu(B_{2r}(x)) \leq 2^{\kappa(R)} \mu(B_r(x)). \quad (1)$$

A function, $g : X \rightarrow [0, \infty]$, is called an *upper gradient* for $f : X \rightarrow \mathbb{R}$, if for all rectifiable curves, $c : [0, \ell] \rightarrow X$, parametrized by arc length, s ,

$$|f(c(\ell)) - f(c(0))| \leq \int_0^\ell g(c(s)) ds. \quad (2)$$

Given an integrable function, $f : B_r(x) \rightarrow \mathbb{R}$, put $\underline{f} = \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f d\mu$.

¹ Our work was motivated by the conjecture of J. Lee and A. Naor, that the Heisenberg group with its Carnot–Caratheodory metric does not bi-Lipschitz embed in L^1 ; see Theorem 6.1. We are very much indebted to Naor for telling us of their conjecture and for explaining its significance in connection with the Goemans–Linial conjecture of theoretical computer science; see Remark 4.

The $(1, p)$ -Poincaré inequality asserts the existence of constants, $\tau(R), \lambda(R)$, such that for all $x \in X, r \leq R$, and all f, g , with g upper gradient for f ,

$$\frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f - \underline{f}| d\mu \leq \tau r \left(\frac{1}{\mu(B_{\lambda r}(x))} \int_{B_{\lambda r}(x)} g^p d\mu \right)^{1/p}. \tag{3}$$

3. Differentiation theory for separable dual space targets

A Banach space V is said to have the *Radon–Nikodym property* if the conclusion of Rademacher’s differentiation theorem holds for every Lipschitz map $\mathbb{R}^k \rightarrow V$. Separable dual spaces and reflexive spaces (separable or not) have the Radon–Nikodym property, [9]. It is difficult to construct spaces with the Radon–Nikodym property which are not isomorphic to a subspace of some separable dual space; see p. 124 and also pp. 121–124 of [2].

An extension of the original Rademacher theorem to PI spaces was obtained in [4]. In [6], we show that the differentiation theory of [4] extends to separable dual space targets.

Let $a(x, x'), b(x, x')$, take values in V and let $\|\cdot\|$ denote the norm of V . The notation,

$$a(x, x') = b(x, x') + o(d^X(x, x')),$$

means

$$\lim_{d^X(x, x') \rightarrow 0} \frac{\|a(x, x') - b(x, x')\|}{d^X(x, x')} = 0.$$

Theorem 3.1. ([6]) *Let X denote a PI space with associated constants κ, λ, τ . Then there exists a countable covering of X by measurable sets, $X = \bigcup_{\alpha} A_{\alpha}$, and Lipschitz maps,*

$$u_{\alpha} : A_{\alpha} \rightarrow \mathbb{R}^{k_{\alpha}},$$

$u_{\alpha} = (u_{1,\alpha}, \dots, u_{k_{\alpha},\alpha})$, with $k_{\alpha} \leq c(\kappa, \tau, \lambda)$, with the following property: For any Lipschitz map, $f : X \rightarrow V$, with V a separable dual space, there exist μ -a.e. unique, bounded measurable functions,

$$\frac{\partial f(x')}{\partial u_{i,\alpha}} : A_{\alpha} \rightarrow V,$$

such that for μ -a.e. $x, x' \in A_{\alpha}$,

$$f(x) = f(x') + \sum_{i=1}^{k_{\alpha}} \frac{\partial f(x')}{\partial u_{i,\alpha}} \cdot (u_{i,\alpha}(x) - u_{i,\alpha}(x')) + o(d^X(x, x')). \tag{4}$$

Remark 1. The countable covering, $X = \bigcup_{\alpha} A_{\alpha}$, and Lipschitz maps, $u_{\alpha} : A_{\alpha} \rightarrow \mathbb{R}^{k_{\alpha}}$, would determine an atlas of charts for a connected Lipschitz manifold, if we added the conditions that k_{α} is independent of α , the measurable sets, A_{α} , are open, and the Lipschitz maps, u_{α} , are bi-Lipschitz onto their images. If in such a case, μ denotes Lebesgue measure, then (4) reduces to the almost everywhere defined first order Taylor expansion of the Lipschitz function, f , in local coordinates u_{α} .

The space, X , also has a *measurable cotangent bundle*, T^*X , which is a measurable vector bundle whose fiber at μ -a.e. $x \in A_{\alpha}$ has dimension k_{α} . Every real valued Lipschitz function, $f : X \rightarrow \mathbb{R}$, has a differential, df , which is a bounded measurable section of T^*X . Likewise, if V is a separable dual space, then any Lipschitz map into V has a differential, which is a section of $T^*X \otimes V$.

Remark 2. An earlier differentiation theory for Lipschitz maps between Carnot groups was developed by Pansu; [15]. For \mathbb{R}^N targets, it coincides with that of [4]. The theory of [15], which uses curve families, has a rather straightforward extension to targets having the Radon–Nikodym property; [6,14]. We do not know if this holds for Theorem 3.1.

4. Bi-Lipschitz nonembedding in separable dual spaces

Generalized differentiation theorems for Lipschitz maps, $f : X \rightarrow V$, can be used to show that under certain conditions on X , f cannot be a bi-Lipschitz embedding. An early such result of Semmes, based on the differentiation theory of Pansu, states that \mathbb{H} does not bi-Lipschitz embed in any finite dimensional Banach space; see [16].

In [4], the differentiation theory developed there was used to prove results on bi-Lipschitz nonembedding for finite dimensional Banach space targets. This unified and extended previously known nonembedding theorems for certain classes of PI spaces, including in particular, Laakso spaces, Carnot groups and Bourdon–Pajot spaces.

In light of the differentiation result, Eq. (4), the bi-Lipschitz nonembedding theorems, Theorems 14.1–14.3 of [4], and their proofs, extend directly to separable dual targets; see, for instance, Theorem 4.1 below.

The doubling condition, (1), together with Gromov’s compactness theorem, implies that any sequence of pointed PI spaces with rescaled metrics, $\{(X, x, c_i \cdot d^X)\}$, where $c_i \rightarrow \infty$, has a subsequence which converges to $(X_x, x_\infty, d_\infty)$ in the pointed measured Gromov–Hausdorff sense. The space X_x is called a *tangent cone* of X at x . Let TX be the *tangent bundle* of X , i.e. the dual of (T^*X) , equipped with the dual norm. Then for μ -a.e. x , there is a *surjective* Lipschitz map $X_x \rightarrow TX_x$; see Section 13 of [4]. Thus,

$$\dim X_x \geq \dim TX_x.$$

Here and below, \dim denotes Hausdorff dimension.

Theorem 4.1. ([6]) *Let X denote a PI space and let $G \subset X$ denote the set of points, $x \in A_\alpha$, such that some tangent cone, X_x , is not bi-Lipschitz to \mathbb{R}^{k_α} . If $\mu(G) > 0$, then X does not bi-Lipschitz embed in any separable dual space.*

The targets covered by Theorem 4.1 include L^p , $1 < p < \infty$, and ℓ^p , $1 \leq p < \infty$.

In view of Remark 2, for the special case, \mathbb{H} , we also have the following generalization of Semmes’ result; see also [14].

Theorem 4.2. ([6]) *A Carnot group does not bi-Lipschitz embed in any Banach space with the Radon–Nikodym property.*

5. Bi-Lipschitz embedding in L^1

The Lipschitz map, $I : (0, \infty) \rightarrow L^1$, given by $I(t) = \chi_{(0,t)}$, is nowhere differentiable, [1]. Thus, L^1 does not have the Radon–Nikodym property. This leads one to ask if the nonembedding result Theorem 4.1 also breaks down when the target is L^1 . Indeed this is the case:

Theorem 5.1. ([8]) *There exist PI spaces, including Laakso spaces, which satisfy the hypothesis of Theorem 4.1, and hence do not bi-Lipschitz embed in any separable dual space, but which do bi-Lipschitz embed in L^1 .*

Remark 3. As far as we know, the PI spaces in Theorem 5.1 provide the first examples of spaces which bi-Lipschitz embed in L^1 but not in ℓ^1 .

6. Generalized differentiation for L^1 and bi-Lipschitz nonembedding

In [7], it is shown that there is a generalized sense in which Lipschitz maps, $\mathbb{R}^n \rightarrow L^1$, or $\mathbb{H} \rightarrow L^1$, are differentiable. As a direct consequence, it follows that \mathbb{H} does not bi-Lipschitz embed in L^1 . This was conjectured by J. Lee and A. Naor.

Theorem 6.1. ([7]) *If $U \subset \mathbb{H}$ is an open subset, and $f : U \rightarrow L^1$ is a Lipschitz map, then for almost every point, $x \in \mathbb{H}$, the map collapses in the direction of the center of \mathbb{H} i.e.*

$$\lim_{g \rightarrow e} \frac{\|f(gx) - f(x)\|_{L^1}}{d^{\mathbb{H}}(gx, x)} = 0, \quad g \in \text{Center}(\mathbb{H}). \quad (5)$$

Corollary 6.2. ([7]) *There is a compact doubling metric space which does not bi-Lipschitz embed in L^1 .*

Let W denote some Cayley graph for the subgroup of the Heisenberg group consisting of matrices with integer entries. Give W the length space metric such that each edge has length 1. For $k \geq 0$, let $W_k := B_k(e) \subset W$ denote the ball of radius k in W , with center the identity element.

Corollary 6.3. ([7]) *The sequence, $\{W_k\}$, is a sequence of uniformly doubling finite graphs with uniformly bounded valence, which do not admit embeddings into L^1 with uniformly bounded bi-Lipschitz distortion.*

Remark 4. A metric space, (Z, ρ) , is said to be of *negative type* if $(Z, \rho^{1/2})$ isometrically embeds in Hilbert space. The Goemans–Linial conjecture asserts that if (Z, ρ) is of negative type, then (Z, ρ) bi-Lipschitz embeds in L^1 . In [14], Lee and Naor show that $(\mathbb{H}, d^{\mathbb{H}})$ is bi-Lipschitz to a metric space of negative type. Therefore, by Theorem 6.1, $(\mathbb{H}, d^{\mathbb{H}})$ is bi-Lipschitz equivalent to a doubling metric space which does not satisfy the conclusion of the Goemans–Linial conjecture. For a prior recent counter example which is not doubling, see [12].

References

- [1] N. Aronszajn, Differentiability of Lipschitzian mappings between Banach spaces, *Studia Math.* 57 (2) (1976) 147–190.
- [2] Y. Benyamini, J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, vol. 1, Amer. Math. Soc. Colloq. Publ., vol. 48, American Mathematical Society, Providence, RI, 2000.
- [3] M. Bourdon, H. Pajot, Poincaré inequalities and quasiconformal structure on the boundary of some hyperbolic buildings, *Proc. Amer. Math. Soc.* 127 (8) (1999) 2315–2324.
- [4] J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces, *Geom. Funct. Anal.* 9 (3) (1999) 428–517.
- [5] J. Cheeger, T. Colding, On the structure of spaces with Ricci curvature bounded below, *J. Differential Geom.* 46 (1997) 406–480.
- [6] J. Cheeger, B. Kleiner, Differentiating Banach space valued functions on metric measure spaces, Preprint, 2005.
- [7] J. Cheeger, B. Kleiner, Differentiating maps to L^1 and the geometry of BV functions, Preprint, 2006.
- [8] J. Cheeger, B. Kleiner, Embedding Laakso spaces in L^1 , Preprint, 2006.
- [9] I.M. Gelfand, Abstrakte Funktionen und lineare Operatoren, *Mat. Sb.* 46 (4) (1938) 235–284.
- [10] M. Gromov, Carnot–Carathéodory spaces seen from within, in: *Sub-Riemannian Geometry*, in: *Progr. in Math.*, Birkhäuser, Basel, 1996, pp. 79–323.
- [11] J. Heinonen, P. Koskela, From local to global in quasiconformal structures, *Proc. Nat. Acad. Sci. USA* 93 (1996) 554–556.
- [12] S. Khot, N. Vishnoi, The unique games conjecture, integrality gap for cut problems and embeddability of negative type metrics into ℓ_1 , in: *46th Annual Symposium on Foundations of Computer Science*, IEE Computer Society, Los Alamitos, 2005, pp. 53–62.
- [13] T.J. Laakso, Ahlfors Q -regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality, *Geom. Funct. Anal.* 10 (1) (2000) 111–123.
- [14] J. Lee, A. Naor, L^p metrics on the Heisenberg group and the Goemans–Linial conjecture, Preprint, 2006.
- [15] P. Pansu, Métriques de Carnot–Carathéodory et quasiisométries des espaces symétriques de rang un, *Ann. of Math.* (2) 129 (1) (1989) 1–60.
- [16] S. Semmes, On the nonexistence of bi-Lipschitz parameterizations and geometric problems about A_∞ -weights, *Rev. Mat. Iberoamericana* 12 (2) (1996) 337–410.