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Lie Algebras/Partial Differential Equations

Generating integrable one dimensional driftless diffusions

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Abstract

A criterion on the diffusion coefficient is formulated that allows the classification of driftless time and state dependent diffusions that are integrable in closed form via point transformations. In the time dependent and state dependent case, a remarkable intertwining with the inhomogeneous Burger's equation is exploited. The criterion is constructive. It allows us to devise families of driftless diffusions parametrized by a rich class of several arbitrary functions for which the solution of any initial value problem can be expressed in closed form. We also derive an elegant form for the master equation for infinitesimal symmetries, previously considered only in the time homogeneous case. ***To cite this article: P. Carr et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Une méthode pour génerer des diffusions intégrables dans le cas unidimensionelle. Nous présentons une condition nécessaire et suffisante sur le coefficient de diffusion $g(x,t)$ d'une diffusion sans drift, afin que celle-ci puisse se réduire, par des transformations ponctuelles des variables dépendentes et indépendantes, à la forme canonique de Lie $u_t - \frac{1}{2}u_{xx} + \frac{A}{x^2}u = 0$ où $A \in \mathbb{R}$. Lie a démontré que celle-ci est la forme canonique d'une diffusion dont le groupe de symétrie est de dimension quatre ou six. Notre résultat complète donc celui de Lie, en donnant une condition locale intrinsèque sur g rendant possible une telle réduction, ainsi qu'une condition constructive, dans la mesure où elle nous permet de construire de façon explicite la solution fondamentale de l'équation correspondante. ***Pour citer cet article : P. Carr et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

Considérons le problème consistant à trouver la probabilité de transition d'une diffusion $\mathrm{d}x_t = g(x_t,t)\,\mathrm{d}W_t$, $t \in [0,T]$ sur un espace de probabilité filtré $(\Omega, \mathcal{B}, P)$, où W_t est un mouvement Brownien uni-dimensionel. Il est bien connu que résoudre ce problème équivaut à déterminer la solution fondamentale de l'équation rétrograde

$$u_t + \frac{1}{2}g^2(x,t)u_{xx} = 0 \quad \text{avec condition finale} \quad u(\xi,T) = \delta_\xi(x). \tag{1}$$

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Un problème de grande importance en physique et en mathématiques financières est de pouvoir exhiber cette solution fondamentale sous une forme explicite. Lie, voulant classifier toutes les équations aux dérivées partielles du second ordre qui puissent se résoudre par un processus « d'intégration », a démontré le théorème suivant :

Proposition 0.1 *(Lie [13]). Soit*

$$\mathcal{L}^{a,b,c}u \equiv u_t + a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u = 0 \tag{2}$$

avec $a(x,t) \neq 0$. L'algèbre de Lie principale $L_{\mathcal{P}}$ (c.a.d. l'algèbre de Lie admise par l'équation (2)*) ayant pour coefficients a, b, c, admet les opérateurs de symétrie triviaux $u\frac{\partial}{\partial u}$ et $\phi(x,t)\frac{\partial}{\partial u}$, où ϕ est une solution de* (2) *et peut se mettre sous la forme*

$$v_\tau = v_{yy} + Z(\tau, y)v \tag{3}$$

par le biais d'une transformation, appelée transformation d'équivalence de Lie, soit :

$$y = \alpha(x,t), \quad \tau = \beta(t), \quad v = \gamma(y,\tau)u(y,\tau), \quad \alpha_x \neq 0, \quad \beta_t \neq 0. \tag{4}$$

Si l'équation (2) *admet une extension de l'algèbre de Lie principale par un opérateur de symétrie supplémentaire, elle se réduit à la forme*

$$v_\tau = v_{yy} + Z(y)v. \tag{5}$$

Si l'algèbre s'étend par trois operateurs supplémentaires (*la partie finie de l'algèbre est de dimension* 4)*, elle se réduit à la forme*

$$v_\tau - v_{yy} + \frac{A}{y^2}v = 0 \quad \text{où } A \text{ est une constante.} \tag{6}$$

Si $\mathcal{L}_p$ s'étend par cinq opérateurs, l'équation (2) *se réduit à l'équation de chaleur*

$$v_\tau - v_{yy} = 0. \tag{7}$$

Notre principal résultat est un critère sur le coefficient de diffusion, qui permet de décider quand une diffusion peut se mettre sous une des formes (6) ou (7). Étant donné que les diffusions considérées peuvent, comme dans le cas des diffusions CEV où $g(x,t) = x^{1+\beta}$, $\beta \in \mathbb{R}$, être dégénérées et que la transformation de Lie–Bluman $y = \int_a^x \frac{1}{g(x',t)}\,\mathrm{d}x' + \zeta(t)$, qui suppose l'intégrabilité de $1/g$, n'est pas dans ces cas-là bien définie, nous introduisons une classe de diffusions dégénérées qui n'est pas la plus générale possible mais qui permet, sans peine, d'appliquer la transformation de Lie–Bluman dans la plupart des cas rencontrés en physique et en mathématiques financières.

Définition 0.2. Soit $I = (l, r) \subset \mathbb{R}$ un intervalle, pouvant être non borné. Soit $\mathcal{L}u = u_t - \frac{1}{2}g^2(x,t)u_{xx} = 0$ une diffusion sur l' intervalle I. Supposons que $g(x,t) \geqslant 0$ soit continu sur I. Définissons de façon itérative un recouvrement fini $\bigcup_i [l_i, r_i] = \bigcup_i I_i = I$ de I avec pour centres associés m_i, selon le procédé suivant :

- Choisissons m_1 avec $g(m_1,t) \neq 0$ et définissons l_1 et r_1 par $l_1 = \inf\{x \in I\colon \int_{m_1}^x \frac{\mathrm{d}x'}{g(x',t)} > -\infty\}$, $r_1 = \sup\{x \in I\colon \int_{m_1}^x \frac{\mathrm{d}x'}{g(x',t)} < +\infty\}$ et posons $R_1^- = \int_{m_1}^{l_1} \frac{\mathrm{d}x'}{g(x',t)}$, $R_1^+ = \int_{m_1}^{r_1} \frac{\mathrm{d}x'}{g(x',t)}$.
- Ayant défini m_i, $R_i^{\pm}$ et I_i pour $i \leqslant i_0$, définissons m_{i_0+1} en choisissant $m_{i_0+1} \in I \setminus \bigcup_{i=1}^{i_0} I_i$ avec $g(m_{i_0+1},t) > 0$ et par la suite en procédant comme ci-dessus.

Définition 0.3. Nous disons qu'une diffusion $\mathcal{L}u = 0$ est *modérémment degénérée* si l_i, r_i et m_i peuvent être choisis indépendemment du temps et si ce recouvrement est fini.

Proposition 0.4. *Soit $\mathcal{L}u = 0$ une diffusion modérémment degénérée avec recouvrement associé $\mathcal{C} = \{I_i = [l_i, r_i]: i = 1,\dots,n\}$. Soit $H^{(i)}(y,t) = \int_{D^{(i)}}^y \tilde{g}^{(i)}(y',t)\,\mathrm{d}y' + m_i$, avec $\tilde{g}^{(i)}$ satisfaisant $\tilde{g}^{(i)}(Y^{(i)}(x,t),t) = g(x,t)$, $y = Y^{(i)}(x,t) = \int_{m_i}^x \frac{1}{g(x',t)}\,\mathrm{d}x' + D^{(i)}(t)$. Pour chaque $I_i \in \mathcal{C}$,*

– *Une condition nécessaire et suffisante afin qu'une diffusion modéremment degénérée puisse se réduire à une forme canonique à quatre dimensions est qu'il existe $\lambda^{(i)} \neq 0$ et des coefficients $A^{(i)}, B^{(i)}, C^{(i)}, D^{(i)}$, tels que $\ddot{D}^{(i)} - 2A^{(i)}D^{(i)} = B^{(i)}$ et tels que pour chaque i, $H^{(i)}$ satisfait l'EDP*

$$H_t - \frac{1}{2}H_{yy} + \beta^{(i)}H_y = 0 \quad \textit{pour } y \in \left(R_i^-, R_i^+\right) \setminus \left\{D^{(i)}(t)\right\}, \tag{8}$$

et les conditions $H(D^{(i)} + R_i^-, t) = l_i$, $H(D^{(i)}, t) = m_i$, $H(D^{(i)} + R_i^+, t) = r_i$ où $\beta^{(i)} = -(\log\alpha^{(i)})_y$, avec $\alpha^{(i)}$ satisfaisant l'équation

$$\alpha_t - \frac{1}{2}\alpha_{yy} + \left(\frac{\lambda^{(i)}}{(y - D^{(i)}(t))^2} + A^{(i)}(t)y^2 + B^{(i)}(t)y + C^{(i)}(t)\right)\alpha = 0 \tag{9}$$

pour $y \in (R_i^-, R_i^+) \setminus \{D^{(i)}(t)\}$. Notons que le produit d'une solution de (8) *et d'une solution de* (9)*, c.à.d. $H^{(i)}\alpha^{(i)}$, satisfait aussi à* (9) *pour $y \neq D^{(i)}(t)$.*

Ce résultat peut être utilisé pour exhiber de nouvelles classes de diffusions dont la solution fondamentale peut s'exprimer sous forme explicite. Une simple extension de la méthode au cas avec drift met en lumière la structure de groupe sous-jacente du résultat de Feller [11] et une extension de certains résultats bien connus pour les procéssus CEV, dans le cas où leur coefficients peuvent dépendre du temps. Voir les exemples dans la version anglaise. Nous remercions Jean Damien Arterit qui nous a aidé à preparer la version française.

1. Introduction

Consider the problem of determining the transition probability of a one-dimensional diffusion $\mathrm{d}x_t = g(x_t, t)\,\mathrm{d}W_t$, $t \in [0, T]$, $x_0 = \xi$, where W_t is a standard Brownian motion with respect to an underlying probability space $(\Omega, \mathcal{B}, P)$. As is well known, under general conditions, this problem is equivalent to finding the solution of the backward Kolmogorov equation with delta function terminal condition for $u(\xi, t, \eta, T)$, i.e. $\frac{\partial u}{\partial t} + \frac{g^2(\xi,t)}{2}\frac{\partial^2 u}{\partial \xi^2} = 0$, $u(\xi, T, \eta, T) = \delta_\eta(\xi)$. In the setting of more general one dimensional diffusions, Sophus Lie [13] discovered a classification of second order differential equations in two variables. A detailed statement of this classification in the parabolic case can be found in the French version of this Note. All diffusion equations possessing a four dimensional symmetry group can be transformed, by appropriate changes of the dependent and independent variables, i.e. Lie's equivalence transformations (see (4)), to the equation (now in forward time) $\frac{\partial v}{\partial t'} - \frac{1}{2}\frac{\partial^2 v}{\partial y^2} + \frac{\lambda}{y^2}v = 0$ where λ is a nonzero constant. The fundamental solution of the above equation can be shown to be (see [12]),

$$F(t, y, \xi) = K\sqrt{\frac{y\xi}{t}}\,\frac{\mathrm{e}^{-\xi y/t}}{\sqrt{t}}\mathcal{I}\left(\kappa - \frac{1}{2}, \frac{\xi y}{t}\right)\exp\left\{-\frac{(y-\xi)^2}{2t}\right\},$$

where $\kappa = \pm\frac{\sqrt{1+8\lambda}}{2}$, K is a normalization constant that ensures $\lim_{t\to 0^+}\int_{\Re} F(t, y, \xi)\,\mathrm{d}\xi = 1$ and $\mathcal{I}(\kappa, \cdot)$ is a modified Bessel function of order κ. It is this class that will be the main focus of the present note which will be devoted to fully characterizing those diffusion coefficients which are associated with the four dimensional symmetry class and will also provide a constructive method for determining such coefficients. In this Note, we concentrate on the important case of a driftless diffusion with no killing. Our results readily extend beyond this case (to diffusions with drift and/or killing) but take on a more complicated and less intuitive form. Although we do not characterize the diffusion coefficients of driftless diffusions with a *two* dimensional symmetry group, we also provide in this Note, for the first time in the time-dependent case, an elegant form for the 'master equation' (see (14)). The latter is a significant step towards finding new classes of driftless diffusions with a two dimensional symmetry group and towards unifying previous special cases.

The financial motivation for the problem addressed in this paper arises when valuing contingent claims written on the price of a single asset following a univariate diffusion. In the PDE to be transformed, $\frac{\partial u}{\partial t} = -\frac{g^2(\xi,t)}{2}\frac{\partial^2 u}{\partial \xi^2}$, t is calendar time, ξ is the relative price of the underlying asset, while u is the relative price of the overlying claim. No arbitrage implies the existence of a probability measure under which all asset prices are martingales, when measured relative to a numeraire. Martingality of the underlying asset price eliminates the term $\frac{\partial u}{\partial \xi}$, while martingality of the overlying claim price eliminates the term $u(\xi,t)$. The function $g(\xi,t)$ relates the instantaneous standard deviation of price changes in the underlying asset to the price level and time. Black Scholes (1973) hypothesized that the underlying volatility $\frac{g(\xi,t)}{\xi}$ is a nonzero constant σ. Under this hypothesis, they showed that the above PDE transforms to the heat equation, $\frac{\partial v}{\partial t'}=\frac{1}{2}\frac{\partial^2 v}{\partial y^2}$. As a result, they valued in closed form a European call paying off $(\xi-K)^+$ at a fixed time $T\geqslant t$. This closed form solution facilitates rapid calculation of theoretical call values, which is useful since calls simultaneously trade for several levels of strike K and maturity T. If the parameter σ is unknown ex ante, then the closed form solution is used to numerically determine the value of σ implied by market prices of European calls. If volatility is in fact constant, then this implied volatility should not depend on the K and T used to infer it. However, observations show that implied volatilities tend to systematically depend on both K and T. These systematic rejections of the constant volatility hypothesis created interest in alternative forms for the function $g(\xi,t)$, which lead to option prices more consistent with data, while retaining the computational benefit afforded by closed form solutions for option prices. For example, one could ask for conditions on $g(\xi,t)$ which still permit transformation to the heat equation. Examples include Bachelier [4]'s original Brownian motion, Cox and Ross [10]'s addition of an absorbing barrier, Rubinstein[15]'s displaced diffusion, and quadratic volatility models described in Albanese et al. [1]. Alternatively, one could ask for conditions on $g(\xi,t)$ which still permit transformation to $\frac{\partial v}{\partial t'}=\frac{1}{2}\frac{\partial^2 v}{\partial y^2}-\frac{\lambda}{y^2}v$. Examples include (i) the Constant Elasticity of Variance (CEV) process of Cox [8], $u_t=-\frac{1}{2}\sigma^2x^{2+2\beta}u_{xx}-(r-q)xu_x$, (ii) Feller's [11] singular diffusion, $u_t=-axu_{xx}-(bx+c)u_x$ on the half line $x\in(0,\infty)$ (4 dimensional except in the cases $c=\frac{a}{2},\frac{3a}{2}$ in which case it is 6 dimensional), where a,b,c are constants, $a>0$, applied to finance in Cox and Ross [10] and in Cox, Ingersoll, and Ross (CIR) [9], (iii) a quadratic drift 3/2 process also proposed in CIR, $(u_t=-\frac{1}{2}\sigma^2x^3u_{xx}-\kappa(\theta-x)xu_x=0$, (iv) Carr and Linetsky's CEV model [6] with jump to default $u_t+\frac{1}{2}a^2(t)x^{2\beta+2}u_{xx}+[\mu(t)x+ca^2(t)x^{2\beta+1}]u_x=[b(t)+ca^2(t)x^{2\beta}]u$ for $\beta<0$. Characterization of the functions g permitting these two kinds of reduction is addressed in the next section.

2. Main results

Introduce the Lie–Bluman transformation (LBT), a *special* Lie equivalence transformation, consisting of a change of independent and a change of dependent variables, that can be used to transform the general diffusion equation (2) to canonical form (3): $y=Y(x,t)=\int_0^x\frac{1}{g(x',t)}\,\mathrm{d}x'+\zeta(t)$, $v=w\exp(\int_\zeta^y[Y_t+\frac{1}{2}\frac{\tilde g_y}{\tilde g}])$ where $\tilde g(y,t)=g(Y^{-1}(x,t),t)$ and where $\zeta(t)$ is an arbitrary function of t. Observe that when applying LBT to a diffusion degenerate at zero (for instance CEV processes mentioned above), there are issues of *non-integrability* for values of $\beta\geqslant0$. That is why below we will introduce a slight modification of the Lie–Bluman methodology that is better suited to deal with such degeneracies. We now are in a position to state our main results. We begin with a definition whose purpose is to split up the domain on which the diffusion is being considered into several maximal subdomains in which LBT can be applied. It is flexible enough to account for most applications found in practice. If the diffusion coefficient is pathological enough, the definition needs to be generalized to include time dependent centers and countably infinite coverings.

Definition 2.1. Let $I=(l,r)\subset\mathbb{R}$ be an open (possibly infinite or semi-infinite) interval. Let $\mathcal{L}u=u_t-\frac{1}{2}g^2(x,t)u_{xx}=0$ be a diffusion on the interval I. Assume that $g(x,t)\geqslant0$ is continuous on I. Define iteratively a finite covering $\bigcup_i[l_i,r_i]=\bigcup_i I_i=I$ of I with collection of centers m_i as follows (with the convention that $[l_i,r_i]=(l,r_i]$ if $l_i=l$ and $[l_i,r_i]=[l_i,r)$ if $r_i=r$ hereafter):

- Choose m_1 where $g(m_1,t)\neq0$ and define l_1, r_1 and then R_1^-, R_1^+ by $l_1=\inf\{x\in I\colon \int_{m_1}^x\frac{\mathrm{d}x'}{g(x',t)}>-\infty\}$, $r_1=\sup\{x\in I\colon \int_{m_1}^x\frac{\mathrm{d}x'}{g(x',t)}<+\infty\}$, $R_1^-=\int_{m_1}^{l_1}\frac{\mathrm{d}x'}{g(x',t)}$, $R_1^+=\int_{m_1}^{r_1}\frac{\mathrm{d}x'}{g(x',t)}$.
- Having defined m_i, $R_i^\pm$ and I_i for $i\leqslant i_0$ define m_{i_0+1} by picking $m_{i_0+1}\in I\setminus\bigcup_{i=1}^{i_0}I_i$ with $g(m_{i_0+1},t)>0$ and proceed as before.

Definition 2.2. We say that $\mathcal{L}u=0$ is a *moderately degenerate diffusion* if the l_i, r_i and m_i's can be chosen independently of time and if the covering is finite.

Proposition 2.3. *Let $\mathcal{L}u = 0$ be a moderately degenerate diffusion as in Definition 2.2 with associated natural covering $\mathcal{C} = \{I_i = [l_i, r_i]:\ i = 1, \dots, n\}$. Let $H^{(i)}(y,t) = \int_{D^{(i)}}^{y} \tilde{g}^{(i)}(y', t)\,\mathrm{d}y' + m_i$, where $\tilde{g}^{(i)}$ satisfies $\tilde{g}^{(i)}(Y^{(i)}(x,t), t) = g(x,t)$ and $y = Y^{(i)}(x,t) = \int_{m_i}^{x} \frac{1}{g(x',t)}\,\mathrm{d}x' + D^{(i)}(t)$. For each $I_i \in \mathcal{C}$,*

(i) *a necessary and sufficient criterion for a moderately degenerate driftless diffusion to be reducible to a four dimensional canonical form is that there exist a $\lambda^{(i)} \neq 0$, coefficients $A^{(i)}, B^{(i)}, C^{(i)}, D^{(i)}$ with $\ddot{D}^{(i)} - 2A^{(i)}D^{(i)} = B^{(i)}$ such that $H^{(i)}$ satisfies the partial differential equation*

$$H_t - \frac{1}{2}H_{yy} + \beta^{(i)} H_y = 0, \quad \text{for } y \in \left(R_i^-, R_i^+\right) \setminus \left\{D^{(i)}(t)\right\}, \tag{10}$$

subject to the conditions (with $l_i \leqslant m_i \leqslant r_i$) $H(D^{(i)} + R_i^-, t) = l_i$, $H(D^{(i)}, t) = m_i$, $H(D^{(i)} + R_i^+, t) = r_i$, where $\beta^{(i)} = -(\log \alpha_2^{(i)})$ and where $\alpha_2^{(i)}$ satisfies the equation

$$\alpha_t - \frac{1}{2}\alpha_{yy} + \left(\frac{\lambda^{(i)}}{(y - D^{(i)}(t))^2} + A^{(i)}(t)y^2 + B^{(i)}(t)y + C^{(i)}(t)\right)\alpha = 0 \tag{11}$$

for $y \in (R_i^-, R_i^+) \setminus \{D^{(i)}(t)\}$. Note that the product of a solution of (10) and a solution of (11), i.e. $\alpha_1^{(i)} := H^{(i)}\alpha_2^{(i)}$, also satisfies (11) when $y \neq D^{(i)}(t)$, hence $H^{(i)}$ is the ratio of two solutions of (11);

(ii) *a necessary and sufficient criterion for a moderately degenerate driftless diffusion to be reducible to a six dimensional canonical form is that $H^{(i)}$ satisfies the partial differential equation*

$$H_t - \frac{1}{2}H_{yy} + \beta^{(i)} H_y = 0, \quad \text{for } y \in \left(R_i^-, R_i^+\right) \setminus \left\{D^{(i)}(t)\right\} \tag{12}$$

subject to the conditions $H(D^{(i)} + R_i^-, t) = l_i$, $H(D^{(i)}, t) = m_i$, $H(D^{(i)} + R_i^+, t) = r_i$, where $\beta^{(i)} = -(\log \alpha^{(i)})_y$ and $\alpha^{(i)}$ satisfies the equation

$$\alpha_t - \frac{1}{2}\alpha_{yy} + \left(A^{(i)}(t)y^2 + B^{(i)}(t)y + C^{(i)}(t)\right)\alpha = 0. \tag{13}$$

Here $A^{(i)}$, $B^{(i)}$ and $C^{(i)}$ are arbitrary time dependent functions. The general solution of Eq. (12) can be expressed as the ratio $\alpha_1^{(i)}/\alpha_2^{(i)}$ of two arbitrary solutions $\alpha_1^{(i)}$ and $\alpha_2^{(i)}$ of Eq. (13) where $\beta^{(i)}$ in (12) is given by $\beta^{(i)} = -(\log \alpha_2^{(i)})_y$.

Remark 1. In the theory of the Schrödinger equations and reduction to Liouville form, a key role is played by the equation $\{z, r\} = 2J(r)$ where $\{z, r\}$ is the Schwarzian derivative. A classical theorem due to Schwarz says that the solution of this equation can be expressed as the ratio of two independent solutions of $\psi''(t) + J(r)\psi = 0$. As far as we can tell, this constructive procedure, in the context of parabolic equations with time dependent coefficients, appears to be new even in the context of reducibility to the heat equation. A related remark, generalizing results in Albanese et al. [1] to the time-dependent case, was made by Albanese [3].

Proposition 2.4 *(Time-independent case). In the special case in which $g(x,t)$ is time-independent, Proposition 2.3 can be put in the following form. Let $v = \log(\tilde{g})_y$.*

(i) *A necessary and sufficient condition for a moderately degenerate driftless diffusion with a time-independent diffusion coefficient to have a four dimensional symmetry group is that $v_y - \frac{1}{2}v^2 = \frac{\lambda}{(y-D)^2} + A(y-D)^2 + C$, where $\lambda \neq 0$, A, C and D are constants.*

(ii) *A necessary and sufficient condition for a moderately degenerate driftless diffusion with a time-independent diffusion coefficient to have a six dimensional symmetry group is that $v_y - \frac{1}{2}v^2 = Ay^2 + By + C$, where A and B and C are constants.*

2.1. Discussion

Note that Proposition 2.3 allows the construction of a rich class of solvable driftless diffusions. This richness is characterized by the freedom to choose the time dependent parameters A, B, C, D in (11) and (13) as well as by the freedom in the choice of initial condition α_1^0 and α_2^0 of the solutions in (11) and (13), all of which can be calibrated to a given implied volatility smile. In the case of the reduction to the heat equation, closely related results to the above have been obtained by Bluman [5]. In the case of time-independent g, there are related results by Albanese and Campolieti [2]. For equations with trivial diffusion term and drift, a complete classification in the four dimensional case was given by Spichak and Stognii [16].

3. Master equation for symmetry group

Following the standard method [14] $X = \tau(t)\partial_t + \xi(x,t)\partial_x + u\hat{\beta}(x,t)\partial_u$ is the ansatz for the infinitesimal generator of the symmetry group. It is easily shown that ξ has the form $\frac{\tau_t(t)}{2} g(x,t)G(x,t) - g(x,t)G_t(x,t)\tau(t) + c(t)$ where $G(x,t) = \int_m^x \frac{1}{g(x',t)}\,\mathrm{d}x'$. One then shows that the master equation for the determination of the symmetry group is of the form: $g(p_{1t} - q_{1x})c - c_{tt} + g(p_{5t} - q_{5x})\tau + g(p_{3t} + p_5 - q_{3x})\tau_t + \frac{1}{2}G\tau_{ttt} = 0$, where $q_i = \frac{1}{2}g^2 p_{ix}$ and where $p_1 = \frac{g_{xx}}{2} - \frac{g_t}{g^2}$, $p_2 = -\frac{1}{g}$, $p_4 = -\frac{1}{2g}G$, $p_3 = -\frac{1}{2}\frac{g_t G}{g^2} + \frac{1}{2}\frac{G_t}{g} + \frac{1}{4}g_{xx}G + \frac{1}{4}\frac{g_x}{g}$, $p_5 = \frac{G_{tt}}{g} + \frac{g_t G_t}{g^2} + \frac{1}{2}\frac{g_{xt}}{g} - \frac{1}{2}g_{xx}G_t$. This generalizes the results of Cicogna and Vitali (see Eq. (14), p. 454 in [7]) to the time dependent case. Next one studies the master equation $\hat{\beta}_{xt} = \hat{\beta}_{tx}$ and finds after a considerable amount of computation and manipulation that it can be cast into the following elegant form

$$
\begin{aligned}
&c\big(U_y - \partial_t\big[\tilde{\beta}\big(D(t),t\big)\big]\big) - (y-D)c_{tt} + \big(U_t + D'U_y - D'''(y-D) + D''D'\big)\tau \\
&\quad + \left(\frac{y-D}{2}U_y + U - \frac{3}{2}D''(y-D) + \frac{(D')^2}{2}\right)\tau_t + \frac{(y-D)^2}{4}\tau_{ttt} = \Sigma(t),
\end{aligned} \tag{14}
$$

where $U = \frac{1}{2}\tilde{\beta}^2 - \frac{1}{2}\tilde{\beta}_y + \partial_t \int_{D(t)}^y \tilde{\beta}\,\mathrm{d}y$, $\tilde{\beta} = -\frac{H_t - \frac{1}{2}H_{yy}}{H_y}$ as in Proposition 2.3 and where $\Sigma(t)$ is an arbitrary function of t. This simple form of the master equation appears to be new, even in the time-independent case.

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