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Some theoretical results on Markov-switching autoregressive models with gamma innovations

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Abstract

In this Note, we give some theoretical results for an original Markov-switching autoregressive model with gamma innovations which has been introduced to describe wind time series. We provide explicit conditions that imply the stability of this model and the consistency of the maximum likelihood estimator. ***To cite this article: P. Ailliot, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Quelques résultats théoriques pour un modèle autorégressif à changements de régimes particulier. Dans cette Note, nous nous intéressons à certaines propriétés théroriques d'un modèle autorégressif à changements de régimes markoviens original qui a été introduit pour décrire des séries temporelles de vent. Dans ce modèle, l'évolution du processus observé dans les différents régimes est paramétrée en utilisant des lois gamma. Nous donnons en particulier des conditions explicites qui impliquent la stabilité de ce modèle ainsi que la convergence des estimateurs du maximum de vraisemblance. ***Pour citer cet article : P. Ailliot, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

1. Introduction

Markov-Switching Autoregressive (MS-AR) models have first been introduced by Hamilton in 1989 [4] to analyse the rate of growth of USA GNP and then used in different fields to model time series subject to discrete regime shifts. A MS-AR process is a bivariate process $\{S_t, Y_t\}$ such that:

– $\{S_t\}$ is an homogeneous Markov chain on a finite space $\mathbf{S} = \{1, \ldots, M\}$. Throughout this Note, this process is assumed to be not observable (or 'hidden'). We will denote $q(i, j; \theta_S) = P(S_t = j | S_{t-1} = i)$ the transition probabilities with $\theta_S = (q(i, j))_{i,j \in \mathbf{S}} \in \boldsymbol{\Theta}_S = \{p(i, j) \geqslant 0, \sum_{j=1}^{M} p(i, j) = 1\}$ the set of parameters which describe the evolution of this process.

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– Conditionally to $\{S_t\}$, $\{Y_t\}$ is a non-homogeneous Markov chain of order $r \geqslant 0$ on $\mathbf{Y} \subset \mathbf{R}^d$. More precisely, we assume that the conditional distribution of Y_t given $\{Y_{t'}\}_{t'<t}$ and $\{S_{t'}\}_{t' \leqslant t}$ only depends on S_t and $\overline{Y}_{t-1} = (Y_{t-r}, \dots, Y_{t-1})$.

Generally, a functional autoregressive model of the form (1) is used to describe the evolution of the observed process $\{Y_t\}$ conditionally to the hidden process $\{S_t\}$:

$$Y_t = f_\theta(\overline{Y}_{t-1}, S_t) + h_\theta(E_t, S_t) \tag{1}$$

where $\{E_t\}$ represents a white noise such that E_t is independent of $Y_{t'}$ for $t' < t$. However, this kind of autoregressive model is not adapted to describe positive time series (e.g., when $\mathbf{Y} = \mathbf{R}^+$) like the wind speed. In this Note, we consider an original MS-AR model which has been proposed for such time series. It has been successfully validated on a wind data set in the North Atlantic (see [1]).

In this model, denoted MS-γAR hereafter, it is assumed that for each $s_t \in \mathbf{S}$, $\bar{y}_{t-1} \in \mathbf{Y}^r$, the conditional distribution $P(Y_t | \overline{Y}_{t-1} = \bar{y}_{t-1}, S_t = s_t)$ is a gamma distribution with mean $\sum_{i=1}^{p} a_i^{(s_t)} y_{t-i} + b^{(s_t)}$ and standard deviation $\sigma^{(s_t)}$. We will denote $g(y_t | \bar{y}_{t-1}; \theta_Y^{(s_t)})$ the density of this conditional distribution with respect to Lebesgue measure with $\theta_Y^{(s)} = (a_1^{(s)}, \dots, a_r^{(s)}, b^{(s)}, \sigma^{(s)}) \in \boldsymbol{\Theta}_\mathbf{Y}$ the parameters which describe the evolution of the observed process in the regime $s \in \mathbf{S}$ and $\boldsymbol{\Theta}_\mathbf{Y}$ a given compact subset of $(\mathbf{R}^+)^r \times \mathbf{R}^{+*} \times \mathbf{R}^{+*}$ (the constraints $a_i^{(s)} \geqslant 0$, $b^{(s)} > 0$ and $\sigma^{(s)} > 0$ are added, for $s \in \mathbf{S}$ and $i \in \{1, \dots, r\}$, in order that the model is well defined). In the sequel, $\theta = (\theta_S, \theta_Y^{(1)}, \dots, \theta_Y^{(M)}) \in \boldsymbol{\Theta}$ will denote the set of unknown parameters, with $\boldsymbol{\Theta} = \boldsymbol{\Theta}_\mathbf{S} \times \boldsymbol{\Theta}_\mathbf{R}^M$. The process $\{X_t\} = \{S_t, \overline{Y}_t\}$ is a first order Markov chain on $\mathbf{S} \times \mathbf{Y}^r$, whose transition kernel will be denoted by Π_θ. Q_θ will be the transition matrix of the hidden chain $\{S_t\}$. In Section 2, we will assume that this transition kernel has a unique invariant distribution and $\overline{P}_\theta^Y$ will denote the stationary law of $\{Y_t\}$.

Theoretical issues concerning MS-AR models have been ignored until recently. In the last few years, several authors have studied the stability of these models and the asymptotic properties of the Maximum Likelihood Estimator (MLE). However, most of the existing results only apply to MS-AR models of the form (1) and MS-γAR models can not be written in this way. In this Note, we propose to adapt some of these results to MS-γAR models. In Section 2, we focus on the asymptotic properties of the MLE and in Section 3 on the stability of these models.

2. Asymptotic properties of the maximum likelihood estimators

In this section, $\{y_t\}_{-r+1 \leqslant t \leqslant T}$ denotes a realization of a MS-γAR process $\{Y_t\}$ with parameter $\theta_0 = (\theta_{S,0}, \theta_{Y,0}^{(1)}, \dots, \theta_{Y,0}^{(M)})$. Following Krishnamurthy et al. [5] and Douc et al. [2], the likelihood we have worked with is the conditional likelihood given $\overline{Y}_0 = \bar{y}_0 \in \mathbf{Y}^r$ and $S_0 = s_0 \in \mathbf{S}$, which is defined as

$$L_{T,s_0}(\theta) = p_\theta(y_1, \dots, y_T | \bar{y}_0, s_0) = \sum_{\{s_1, \dots, s_T\} \in \mathbf{S}^T} p_\theta(y_1, \dots, y_T, s_1, \dots, s_T | \bar{y}_0, s_0) \tag{2}$$

with $p_\theta(y_1, \dots, y_T, s_1, \dots, s_T | \bar{y}_0, s_0) = \prod_{t=1}^{T} q(s_{t-1}, s_t; \theta_S) g(y_t | \bar{y}_{t-1}, \theta_Y^{(s_t)})$. A maximum point of L_{T,s_0} will be denoted by $\hat{\theta}_{T,s_0}$ and abusively called a MLE. We will see below that, under general assumptions, the choice of s_0 does not affect the asymptotic properties of the MLE.

Whereas the numerical computation of the MLE in models with hidden variables has been addressed by many authors (see Ailliot [1] and references therein), the statistical issues concerning the asymptotic properties of these estimators have been addressed only recently. The proof of the consistency of the MLE for general MS-AR models has been established simultaneously in 1998 by Francq et al. [3] and Krishnamurthy et al. [5]. These general results can be used in order to establish the consistency theorem given below. Let us first define the relation $\backsim$ on $\boldsymbol{\Theta}$ by $\theta_1 \backsim \theta_2$ if and only if θ_1 and θ_2 define the same model, up to a permutation of indices.

Theorem 2.1. *Assume that conditions* **(C1)** *and* **(C2)** *below hold.*

(C1) *(Stability): For all $\theta \in \boldsymbol{\Theta}$ the stochastic matrix Q_θ is irreducible and the transition kernel Π_θ admits a unique stationary distribution with a moment of order $\kappa > 2$.*

(C2) *(Identifiability): $\theta_{R,0}^{(s)} \neq \theta_{R,0}^{(s')}$ if $s \neq s'$.*

Then, for all $s_0 \in \mathbf{S}$, $\hat{\theta}_{T,s_0} \to \theta_0$ $\overline{P}^Y_{\theta_0}$-a.s. when $T \to \infty$, where the convergence is interpreted in the quotient topology generated by $\backsim$.

The proof of this theorem consists in demonstrating that the general assumptions given in [5] are verified by MS-γAR models if conditions **(C1)** and **(C2)** hold. At first, we must demonstrate that $\overline{P}^Y_{\theta_0} = \overline{P}^Y_{\theta}$ if and only if $\theta_0 \backsim \theta$, that is the identifiability of the parameter θ_0 as defined in [2]. Then, we have to prove that the emission densities $g(y_t|\bar{y}_{t-1}; \theta_Y^{(s)})$ satisfy some continuity and existence of moment conditions under **(C2)**. A detailed proof of these results is given in [1].

Assumptions which warrant consistency and asymptotic normality of the MLE in MS-AR models with compact, not necessary finite, hidden state space **S** can be found in [2], and as far as we know, this is the only existing result about asymptotic normality of MLE in MS-AR models. In this paper it is assumed that the transition kernel $g(y_1|\bar{y}_0; \theta_Y)$ and its derivatives are bounded (see assumptions (A1)(b) and (A8)(b) in [2]) and these conditions do not hold for MS-γAR models, except if we impose unrealistic constraints on the parameters. However, we may argue that these conditions can be weakened in the particular case when the hidden state-space **S** is finite. Indeed, in this particular case, certain integrals becomes finite sums, and it becomes possible to exchange limits and derivatives with integrals without the assumptions (A1)(b) and (A8)(b). Such weaker conditions are proposed in [1]. If we admit that they are sufficient, the asymptotic normality of the MLE is implied by **(C2)** and the condition **(C3)** below:

(C3) *(Stability)*: For all $\theta \in \boldsymbol{\Theta}$ the stochastic matrix Q_θ is positive and the transition kernel Π_θ admits a unique stationary distribution with a moment of order $\kappa > 4$.

3. Stability

The results given in the previous section are not fully satisfactory if we do not provide explicit criteria which imply the stability conditions **(C1)** and **(C3)**. This problem is discussed in this section.

Theorem 3.1. *Let $\{X_t\} = \{S_t, Y_t\}$ be a MS-γAR process of order $r = 1$. Assume that the Markov chain $\{S_t\}$ is irreducible and aperiodic, and let $\mu = (\mu_1, \ldots, \mu_M)$ be its stationary distribution. Then if $\sum_{s\in\mathbf{S}} \pi_s \log(a_1^{(s)}) < 0$, $\{X_t\}$ is geometrically ergodic, and in particular the transition kernel Π_θ admits a unique invariant distribution.*

The proof of this theorem is close to the one given in Yao et al. [7], who have studied the stability of functional MS-AR models. The main difficulty consists in demonstrating that the transition kernel Π_θ satisfies a drift condition. The detailed proof of this result can be found in [1].

The geometric ergodicity of the kernel Π_θ is not sufficient to prove the consistency of the MLE, and we need conditions that ensure existence of moments for the stationary distribution. Such conditions are given in the Theorem 3.2 below.

Theorem 3.2. *Let $\{X_t\} = \{S_t, Y_t\}$ be a MS-γAR process of order $r = 1$. Assume that the Markov chain $\{S_t\}$ is irreducible and aperiodic and that the spectral radius of the matrix $R_\kappa = (q(i,j)(a_1^{(j)})^\kappa)_{i,j\in\mathbf{S}}$ is less than 1 for some $\kappa \geqslant 1$. Then $\{X_t\}$ is geometrically ergodic and the stationary distribution of $\{Y_t\}$ admits a moment of order κ.*

Proof of Theorem 3.2. The conclusion of this theorem follows from the facts that the Markov chain $\{X_t\}$ satisfies the drift condition (3) given below, is irreducible and has the strong Feller property (see [1,6,7]).

$$\exists A < 1, \exists B \in \mathbf{R}, \forall (s_0, y_0) \in \mathbf{S} \times \mathbf{Y} \quad E\big[(Y_t)^\kappa | s_0, y_0\big] \leqslant A \times (y_0)^\kappa + B. \tag{3}$$

The irreducibility of $\{X_t\}$ comes from the one of $\{S_t\}$ and the fact that $g(y_1|y_0; \theta_Y^{(s)}) > 0$ $\forall s, y_0, y_1 \in \mathbf{S} \times \mathbf{Y}^2$. We can check that it has the strong Feller property using the dominated convergence theorem. Let us now briefly explain how we have obtained (3). By definition, we have

$$E\big[(Y_t)^\kappa | S_0, \ldots, S_t, Y_0, \ldots, Y_{t-1}\big] = \gamma_\kappa\big(a^{(S_t)} Y_{t-1} + b^{(S_t)}, \sigma^{(S_t)}\big)$$

with $\gamma_\kappa(\mu,\sigma)$ the moment of order κ of the gamma distribution with mean μ and standard deviation σ. We can check, using asymptotic expansion of the gamma function, that $\gamma_\kappa(\mu,\sigma)=\mu^\kappa(1-\frac{\kappa(\kappa+1)\sigma^2}{2\mu^2}+o_\infty(\frac{1}{\mu^2}))$, and then that there exit constants K and L such that $\forall s_0, y_0 \in \mathbf{S}\times\mathbf{Y}$

$$E\big[(Y_t)^\kappa|s_0,y_0\big]\leqslant E\big[\big(a^{(S_t)}\cdots a^{(S_1)}\big)^\kappa|s_0\big]\times(y_0)^\kappa+K\times(y_0)^{\kappa-1}+L.$$

Then, we can use the same arguments as in the Theorem 2 of [7] to show that, if the spectral radius of the matrix R_κ is less than 1 for some $\kappa\geqslant 1$, then there exists $A<1$ and $T\in\mathbf{N}$ such that $\forall s_0\in\mathbf{S}$ $E[(a^{(S_t)}\cdots a^{(S_1)})^\kappa|s_0]<A$. Finally, we deduce that the drift condition (3) holds. □

From this theorem, we can deduce explicit conditions on the parameter θ_0 which imply the consistency and the asymptotic normality of the MLE. The results given in this section are valid only for models of order $r=1$. Extensions to models of higher order are discussed in [1], but the proposed conditions are probably not optimal. The improvement of these conditions is currently further investigated.

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