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Markov selections and their regularity for the three-dimensional stochastic Navier–Stokes equations

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Abstract

The martingale problem associated to the three-dimensional Navier–Stokes equations is shown to have a family of solutions satisfying the Markov property. The result is achieved by means of an abstract selection principle. The Markov property is crucial to extend the regularity of the transition semigroup from small times to arbitrary times, thus showing that *every* Markov selection has a property of continuous dependence on initial conditions. ***To cite this article: F. Flandoli, M. Romito, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Sélections Markoviennes et leur régularité pour les équations stochastiques de Navier–Stokes tridimensionnelles. Il est établi que le problème de martingales associé aux équations de Navier–Stokes tridimensionnelles possède une famille de solutions qui satisfont la propriété de Markov. Ce résultat est obtenu par un principe abstrait de sélection. La propriété de Markov est fondamentale pour étendre la régularité du semi groupe de transition des petites échelles de temps à des échelles arbitraires, en établissant en particulier que *chaque* sélection de Markov dépend continûment des conditions initiales. ***Pour citer cet article : F. Flandoli, M. Romito, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

1. Introduction

Uniqueness of weak solutions and global existence of smooth solutions are known to be the most important open problems related to the analysis of the Navier–Stokes equations. Another open problem is the continuous dependence of solutions on initial data. Such problems are also related to each other.

The purpose of this Note is to present a way of showing that there is continuous dependence with respect to initial data for the stochastic Navier–Stokes equations, for all Markov solutions, under certain non-degeneracy assumptions on the noise. We consider the following equations on the torus $[0, 1]^3$,

$$du + (u \cdot \nabla)u\,dt + \nabla p\,dt = \nu \Delta u\,dt + \mathcal{Q}^{1/2}\,dW, \tag{1}$$

with $\operatorname{div} u = 0$ and with periodic boundary conditions (details on the equations are given below).

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It is well known that there is at least one solution to the above equations, for any initial condition with finite energy, but, as in the deterministic case, it is still open if the solution is unique in law. Whenever it holds, uniqueness in law implies additional features, such as the Markov property. We use an abstract Markov selection principle to show existence of solutions to the martingale problem associated to (1) with the Markov property. The selection principle presented here is a generalisation to infinite dimensions of a result of Krylov [5] for SDE on a finite dimensional state space (see also Stroock and Varadhan [6]).

Due to the lack of continuity of trajectories of the solutions, the Markov property for selections holds only for almost every time (see later Definition 2.1). Moreover, the map *initial data* $\to$ *solution* is only measurable. On the other hand, since for small times and regular initial conditions the equations have a nice behaviour, one can get estimates on the dependence with respect to initial data. If the noise is sufficiently non-degenerate, the law of the solution at a small time t depends continuously on initial data in the variational norm (up to a small error). The Markov property then allows us to extend these estimates to all times, so that any Markov selection has a Strong Feller property (measured in the topology of the domain of powers of the Laplace operator).

Under strong non-degeneracy assumptions on the noise, the existence of a particular selection generating a Markov process semigroup and having suitable strong Feller regularity in the initial condition has been proved by Da Prato and Debussche [1] (and refined recently to be a Markov process in Debussche and Odasso [2]). Our approach is entirely different. We prove existence of a Markov selection without any restriction on the noise, by means of an abstract selection principle. Then, under non-degeneracy assumptions on the noise that extend [1], we prove that *every* Markov selection is regular in the initial condition. Our proof is entirely different as well, and quite more direct.

A more detailed account of this work, with complete proofs, is given in a companion paper [4].

2. The martingale problem for the Navier–Stokes equations

Let $\mathcal{D}^\infty$ be the space of infinitely differentiable divergence-free periodic vector fields on $\mathcal{T} = [0,1]^3$, with zero mean, and denote by H, V, respectively, the closure of $\mathcal{D}^\infty$ with respect to the L^2 and the H^1 norm. The equations are written in the abstract form $\mathrm{d}u + [Au + B(u,u)]\,\mathrm{d}t = \mathcal{Q}^{1/2}\,\mathrm{d}W$ by projecting (*Leray's projection*) Eqs. (1) onto the space of divergence-free vector fields. Here, A denotes the Stokes operator (the realisation of the Laplace operator on H), $B : V \times V \to V'$ is the projection of the Navier–Stokes non-linearity onto the dual space V' of V, $\mathcal{Q} : H \to H$ is a symmetric non-negative trace-class operator, with trace denoted by $\sigma^2 = \operatorname{Tr}\mathcal{Q}$, and W is a cylindrical Wiener process on H. Finally, let $\Omega = C([0,\infty); D(A)')$, $\mathcal{B}$ the Borel sets of Ω, and define on Ω the (natural) filtration $\mathcal{B}_t = \sigma(\xi_s\colon 0 \leqslant s \leqslant t)$, where ξ is the canonical process on Ω.

In order to give a definition (Definition 2.2 below) of solutions to the Navier–Stokes equations which is compatible with both the conditional structure of Markov processes and the peculiar regularity properties of the equations (notice that the energy inequality on a time interval $[s,t]$ is known to hold only for a.e. s), we introduce the following definitions:

Definition 2.1. An adapted process $(\theta_t, \mathcal{B}_t, P)_{t\geqslant 0}$ on Ω is a *a.s. super-martingale* if it is P-integrable for all $t \geqslant 0$ and there is a set $T_\theta \subset (0,\infty)$ with null Lebesgue measure, such that for all $s \notin T_\theta$ and all $t > s$,

$$\mathbb{E}^P[\theta_t | \mathcal{B}_s] \leqslant \theta_s.$$

Definition 2.2. Given a probability measure μ_0 on H, a probability P on $(\Omega, \mathcal{B})$ is a solution starting at μ_0 to the martingale problem associated to the Navier–Stokes equations (1) if

(i) $P[L^\infty_{\mathrm{loc}}([0,\infty); H) \cap L^2_{\mathrm{loc}}([0,\infty); V)] = 1$;

(ii) for each $\varphi \in \mathcal{D}^\infty$ the process $(M^\varphi_t, \mathcal{B}_t, P)$, defined P-a.s. on $(\Omega, \mathcal{B})$ as $M^\varphi_t = \langle \xi_t - \xi_0, \varphi\rangle + \nu\int_0^t \langle \xi_s, A\varphi\rangle\,\mathrm{d}s - \int_0^t \langle B(\xi_s,\varphi), \xi_s\rangle\,\mathrm{d}s$ is a square integrable continuous martingale with quadratic variation $[M^\varphi]_t = t|\mathcal{Q}^{1/2}\varphi|^2_H$;

(iii) for each $n \geqslant 1$, the process E^n_t, defined P-a.s. on $(\Omega, \mathcal{B})$ as

$$E^n_t = |\xi_t|^{2n}_H + 2n\nu \int_0^t |\xi_s|^{2n-2}_H |\xi_s|^2_V\,\mathrm{d}s - |\xi_0|^{2n}_H - n(2n-1)\sigma^2 \int_0^t |\xi_s|^{2n-2}_H\,\mathrm{d}s$$

is P-integrable and $(E^n_t, \mathcal{B}_t, P)$ is an a.s. super-martingale;

(iv) the marginal of P at time $t = 0$ is μ_0.

3. Existence of Markov selections

The following existence theorem of a Markov solution to (1) relies essentially on Definition 2.2 above:

Theorem 3.1. *There exists a family $(P_x)_{x\in H}$ of probability measures on $(\Omega,\mathcal{B})$ such that for each $x \in H$, P_x is a solution to the martingale problem with initial distribution δ_x, and there is a set $T \subset (0,\infty)$ with null Lebesgue measure such that for all $s \notin T$, all $t \geqslant s$ and all bounded measurable $\varphi : H \to \mathbf{R}$, $\mathbb{E}^{P_x}[\varphi(\xi_t)|\mathcal{B}_s] = \mathbb{E}^{P_{\xi_s}}[\varphi(\xi_{t-s})]$.*

The proof of this result is based on an abstract selection principle and we give a few details: define for each $x \in H$ the set $\mathcal{C}(x) = \{P\colon\ P$ *is a solution, starting at x, of the martingale problem*$\}$. The first step of the proof is to show that the family of sets $(\mathcal{C}(x))_{x\in H}$ fulfils the following properties:

(i) each set $\mathcal{C}(x)$ is non-empty and compact in the space of probability measures on Ω, and $x \in H \mapsto \mathcal{C}(x)$ is measurable;
(ii) for each $x \in H$ and for all $P \in \mathcal{C}(x)$, $P[C([0,\infty); H_\sigma)] = 1$, where H_σ denotes the space H endowed with the weak topology;
(iii) there is a set $T \subset (0,\infty)$ of null Lebesgue measure such that for all $t \notin T$, $x \in H$ and $P \in \mathcal{C}(x)$,
   (a) there exists $N \in \mathcal{B}_t$, with $P[N] = 0$ such that for all $\omega \notin N$, $\omega(t) \in H$ and P_t^ω is a solution to the martingale problem starting at $\omega(t)$ (suitably translated in time), where $(P_t^\omega)_{\omega\in\Omega}$ is a regular conditional probability distribution of P, given $\mathcal{B}_t$,
   (b) for each $\mathcal{B}_t$-measurable map $\omega \in \Omega \mapsto Q_\omega$ such that there is $N \in \mathcal{B}_t$ with $P[N] = 0$ and for all $\omega \notin N$, $\omega(t) \in H$ and Q_ω is a solution to the martingale problem starting at $\omega(t)$, then the probability $P^Q \in \mathcal{C}(x)$ (the measure P^Q is defined, roughly, as P until time t, and as Q_ω, conditional to $\mathcal{B}_t$, after time t).

A generalisation to Hilbert spaces of the abstract selection principle of Stroock and Varadhan [6], then allows to select, from the multi-valued map $x \mapsto \mathcal{C}(x)$, a measurable selection. The solutions are chosen using consecutive optimisations (that preserve the above properties) of suitable functionals on the space of probability measures on Ω. Hence, the selection itself fulfils the three properties and in particular the last one. This last fact implies that the almost sure Markovianity holds.

4. Regularity of Markov selections

We assume further that the covariance operator is given as $\mathcal{Q}^{1/2} = A^{-3/4-\alpha_0}\widetilde{\mathcal{Q}}^{1/2}$, for some $\alpha_0 > 0$, where $\widetilde{\mathcal{Q}}$ is an isomorphism of H. Under this assumption, $(\mathcal{Q}^{1/2}W_t)_{t\geqslant 0}$ is a Brownian motion in $D(A^\alpha)$, for all $\alpha < \alpha_0$. We also set $\mathcal{W} = D(A^{\theta_0})$, where $\theta_0 = \frac{1}{2}(\alpha_0+1)$ if $\alpha_0 \in (0,\frac{1}{2})$ and $\theta_0 = \alpha_0 + \frac{1}{4}$ if $\alpha_0 > \frac{1}{2}$.

Given an a.s. Markov process $(P_x)_{x\in H}$ we associate to it the operators $\mathcal{P}_t$ on $B_b(H)$ defined as $(\mathcal{P}_t\varphi)(x) = \mathbb{E}^{P_x}[\varphi(\xi_t)]$. Such operators do not form a semigroup since $\mathcal{P}_{t+s} = \mathcal{P}_t\mathcal{P}_s$ holds for every $t \geqslant 0$ and *almost* every $s \geqslant 0$.

Definition 4.1. Given $\epsilon > 0$, $x_0 \in \mathcal{W}$ and $t_0 > 0$, we say that $(\mathcal{P}_t)_{t\geqslant 0}$ is $\mathcal{W}$-*Strong Feller* at (t_0, x_0) up to the error ϵ if for every $\varepsilon > 0$ there is $\delta = \delta_{x_0,t_0,\epsilon,\varepsilon} > 0$ such that $|(\mathcal{P}_{t_0}\psi)(x_0+h) - (\mathcal{P}_{t_0}\psi)(x_0)| \leqslant \epsilon + \varepsilon$, for every $\psi \in B_b(H)$ with $|\psi|_\infty \leqslant 1$, and for every $h \in \mathcal{W}$ with $|h|_\mathcal{W} < \delta$. If we can choose $\epsilon = 0$ in the previous condition, we simply say that $(\mathcal{P}_t)_{t\geqslant 0}$ is $\mathcal{W}$-Strong Feller at (t_0, x_0) ("without error").

In general, we say that $(\mathcal{P}_t)_{t\geqslant 0}$ is $\mathcal{W}$-Strong Feller at $T > 0$ if $\mathcal{P}_T\psi \in C_b(\mathcal{W})$ for every $\psi \in B_b(H)$. An important detail of the previous definition is the *uniformity* in $\psi \in B_b(H)$, $|\psi|_\infty \leqslant 1$. This is the tool to transfer the continuity property from small to arbitrary times: see the following lemma.

Lemma 4.2. *Given $T > 0$, assume that for every $\epsilon > 0$ and $x_0 \in \mathcal{W}$ there is $t_0 \in (0,T)$ such that $(\mathcal{P}_t)_{t\geqslant 0}$ is $\mathcal{W}$-Strong Feller at (t_0,x_0) up to the error ϵ. Then $(\mathcal{P}_t)_{t\geqslant 0}$ is $\mathcal{W}$-Strong Feller at time T.*

Proof. The proof is based on rewriting $|(\mathcal{P}_T\psi)(x_0+h) - (\mathcal{P}_T\psi)(x_0)|$ as

$$\big|(\mathcal{P}_{t_0+s}\mathcal{P}_{T-t_0-s}\psi)(x_0+h) - (\mathcal{P}_{t_0+s}\mathcal{P}_{T-t_0-s}\psi)(x_0)\big|.$$

One can choose (recall we assume the Markov property only a.s. in one of the arguments) a value of s such that this decomposition holds and $\mathcal{P}_{t_0+s}$ has a property similar to $\mathcal{P}_{t_0}$. □

Lemma 4.3. *Assume we have two families of operators* $(\mathcal{P}_t)_{t\geqslant 0}$ *and* $(\mathcal{P}_t^{(R)})_{t\geqslant 0}$ *as above, indexed by* $R > 0$. *Assume that, given* $\epsilon > 0$ *and* $x_0 \in \mathcal{W}$, *there are* (t_0, R_0), *with the possibility to choose* t_0 *arbitrarily small, such that*:

(i) $|(\mathcal{P}_{t_0}^{(R_0)}\varphi)(x_0+h) - (\mathcal{P}_{t_0}\varphi)(x_0+h)| \leqslant \frac{\epsilon}{2}$ *for every* $\varphi \in B_b(H)$, $|\varphi|_\infty \leqslant 1$, *and every* $h \in \mathcal{W}$, $|h|_{\mathcal{W}} < 1$;
(ii) $(\mathcal{P}_t^{(R_0)})_{t\geqslant 0}$ *is* $\mathcal{W}$*-Strong Feller at* (t_0, x_0) (“*without error*”).

Then $(\mathcal{P}_t)_{t\geqslant 0}$ *is* $\mathcal{W}$*-Strong Feller for all positive times.*

The proof of the above lemma is obvious, from triangle inequality and Lemma 4.2. We are now able to prove the main continuity theorem.

Theorem 4.4. *Let* $(P_x)_{x\in H}$ *be any a.s. Markov process associated to the Navier–Stokes equations* (1) *and let* $(\mathcal{P}_t)_{t\geqslant 0}$ *be the operators on* $B_b(H)$ *defined as above. Then* $(\mathcal{P}_t)_{t\geqslant 0}$ *is* $\mathcal{W}$*-Strong Feller for all times.*

Proof. We consider the equation $\mathrm{d}u + [Au + B(u,u)\chi_R(|u|_{\mathcal{W}}^2)]\,\mathrm{d}t = \mathcal{Q}^{1/2}\,\mathrm{d}W$, with initial condition $u(0) = x$, where $\chi_R : [0, +\infty) \to [0, 1]$ is a non-increasing smooth function, equal to 1 on $[0, R]$, to 0 on $[R+2, \infty)$, and with derivative bounded by 1. One can show that this equation is well-posed (the details are classical); moreover, if $x \in \mathcal{W}$, the solution has continuous paths in $\mathcal{W}$ and coincides with any weak solution of the original equation up to a certain strictly positive random time τ_R^x, defined as the first time the locally unique and regular solution u^x of the original equation (recall that $x \in \mathcal{W}$) has the property $|u^x(t)|_{\mathcal{W}} = R$. Denote by $\mathcal{P}_t^{(R)}$ the associated Markov semigroup. It is not difficult to check that $|(\mathcal{P}_t^{(R)}\varphi)(x) - (\mathcal{P}_t\varphi)(x)| \leqslant 2|\varphi|_\infty P_x[\sup_{s\in[0,t]} |\xi_s|_{\mathcal{W}} \geqslant R] = 2|\varphi|_\infty P_x[\tau_R^x \leqslant t]$. Since, for $x \in \mathcal{W}$, $\tau_R^x > 0$ a.s., one can deduce that condition (i) of Lemma 4.3 is true. Finally, condition (ii) of Lemma 4.3 is a classical result that can be proved using the Bismuth–Elworthy–Li formula, as in Flandoli and Maslowski [3], Da Prato and Debussche [1] and other references. This completes the proof. □

As an example of application (others can be found in Flandoli and Romito [4]) of the above results, we state a condition for well-posedness.

Theorem 4.5. *Assume that there are* $x_0 \in \mathcal{W}$, $t_0 > 0$ *and a solution* $\widetilde{P}_{x_0}$ *of the martingale problem such that* $\widetilde{P}_{x_0}[C([0, t_0]; \mathcal{W})] = 1$. *Then, for every Markov selection* $(P_x)_{x\in H}$ *we have* $P_x[C([0, \infty; \mathcal{W})] = 1$ *for all* $x \in \mathcal{W}$. *In particular, path-wise uniqueness holds for every* $x \in \mathcal{W}$.

Proof. We just give a sketch. The assumption implies, by the Markov property and non-degeneracy of the noise, that $P_x[C([0, t_0); \mathcal{W})] = 1$ for all x in a dense set of $\mathcal{W}$, hence for all $x \in \mathcal{W}$ by the strong Feller property. By the Markov property, again, this extends to all times. □

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