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Algebraic Geometry

On the irreducibility of Deligne–Lusztig varieties

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Abstract

Let $\mathbf{G}$ be a connected reductive algebraic group defined over an algebraic closure of a finite field and let $F:\mathbf{G}\to\mathbf{G}$ be an endomorphism such that F^δ is a Frobenius endomorphism for some $\delta \geqslant 1$. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$. We prove that the Deligne–Lusztig variety $\{g\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P}\cdot F(\mathbf{P})\}$ is irreducible if and only if $\mathbf{P}$ is not contained in a proper F-stable parabolic subgroup of $\mathbf{G}$. ***To cite this article: C. Bonnafé, R. Rouquier, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Sur l'irréductibilité des variétés de Deligne–Lusztig. Soit $\mathbf{G}$ un groupe réductif connexe défini sur une clôture algébrique d'un corps fini et soit $F:\mathbf{G}\to\mathbf{G}$ un endomorphisme dont une puissance est un endomorphisme de Frobenius. Soit $\mathbf{P}$ un sous-groupe parabolique de $\mathbf{G}$. Nous montrons que la variété de Deligne–Lusztig $\{g\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P}\cdot F(\mathbf{P})\}$ est irréductible si et seulement si $\mathbf{P}$ n'est pas contenu dans un sous-groupe parabolique F-stable propre de $\mathbf{G}$. ***Pour citer cet article : C. Bonnafé, R. Rouquier, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Let $\mathbf{G}$ be a connected reductive group over an algebraic closure of a finite field and let $F:\mathbf{G}\to\mathbf{G}$ be an endomorphism such that some power of F is a Frobenius endomorphism of $\mathbf{G}$. Let $\mathcal{L}:\mathbf{G}\to\mathbf{G}$, $g\mapsto g^{-1}F(g)$ be the Lang map. It is surjective and étale. If $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$, we set

$$\mathbf{X}_\mathbf{P} = \{g\mathbf{P}\in\mathbf{G}/\mathbf{P} \mid \mathcal{L}(g)\in\mathbf{P}\cdot F(\mathbf{P})\}.$$

This is the Deligne–Lusztig variety associated to $\mathbf{P}$. The aim of this Note is to prove the following result:

Theorem 1. *Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$. Then $\mathbf{X}_\mathbf{P}$ is irreducible if and only if $\mathbf{P}$ is not contained in a proper F-stable parabolic subgroup of $\mathbf{G}$.*

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Note that this result has been obtained independently by Lusztig (unpublished) and Digne and Michel [2, Proposition 8.4] in the case where $\mathbf{P}$ is a Borel subgroup: both proofs are obtained by counting rational points. We present here a geometric proof (inspired by an argument of Deligne [3, proof of Proposition 4.8]) which reduces the problem to the irreducibility of the Deligne–Lusztig variety associated to a Coxeter element: this case has been treated by Deligne and Lusztig [3, Proposition 4.8].

Before starting the proof of this theorem, we first describe an equivalent statement. Let $\mathbf{B}$ be an F-stable Borel subgroup of $\mathbf{G}$, let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{B}$, let W be the Weyl group of $\mathbf{G}$ relative to $\mathbf{T}$ and let S be the set of simple reflections of W with respect to $\mathbf{B}$. We denote again by F the automorphism of W induced by F. Given $I \subset S$, let W_I denote the standard parabolic subgroup of W generated by I and let $\mathbf{P}_I = \mathbf{B}W_I\mathbf{B}$. We denote by $\mathcal{P}_I$ the variety of parabolic subgroups of $\mathbf{G}$ of type I (i.e. conjugate to $\mathbf{P}_I$) and by $\mathcal{B}$ the variety of Borel subgroups of $\mathbf{G}$ (i.e. $\mathcal{B} = \mathcal{P}_\varnothing$). For $w \in W$, we denote by $\mathcal{O}_I(w)$ the $\mathbf{G}$-orbit of $(\mathbf{P}_I, {}^w\mathbf{P}_{F(I)})$ in $\mathcal{P}_I \times \mathcal{P}_{F(I)}$. Note that $\mathcal{O}_I(w)$ depends only on the double coset $W_I w W_{F(I)}$. We define now

$$\mathbf{X}_I(w) = \big\{\mathbf{P} \in \mathcal{P}_I \mid (\mathbf{P}, F(\mathbf{P})) \in \mathcal{O}_I(w)\big\}.$$

The group $\mathbf{G}^F$ acts on $\mathbf{X}_I(w)$ by conjugation. We set $\mathcal{O}(w) = \mathcal{O}_\varnothing(w)$ and $\mathbf{X}(w) = \mathbf{X}_\varnothing(w)$.

Theorem 2. *Let $I \subset S$ and let $w \in W$. Then $\mathbf{X}_I(w)$ is irreducible if and only if $W_I w$ is not contained in a proper F-stable standard parabolic subgroup of W.*

Remark 1. Let us explain why Theorems 1 and 2 are equivalent. Let $\mathbf{P}_0$ be a parabolic subgroup of $\mathbf{G}$. Let I be its type and let $g_0 \in \mathbf{G}$ be such that $\mathbf{P}_0 = {}^{g_0}\mathbf{P}_I$. Let $w \in W$ be such that $\mathcal{L}(g_0) \in \mathbf{P}_I w \mathbf{P}_{F(I)}$. The pair $(I, W_I w W_{F(I)})$ is uniquely determined by $\mathbf{P}_0$. Then, the map $\mathbf{X}_{\mathbf{P}_0} \to \mathbf{X}_I(w)$, $g\mathbf{P}_0 \mapsto {}^{gg_0}\mathbf{P}_I$ is an isomorphism of varieties (indeed, it is straightforward that $\mathcal{L}(g) \in \mathbf{P}_0 \cdot F(\mathbf{P}_0)$ if and only if $\mathcal{L}(gg_0) \in \mathbf{P}_I w \mathbf{P}_{F(I)}$).

Let $\mathbf{Q}$ be a parabolic subgroup of $\mathbf{G}$ containing $\mathbf{P}$. Let J be its type. Then $I \subset J$, $\mathbf{Q} = {}^{g_0}\mathbf{P}_J$ and $\mathcal{L}(g_0) \in \mathbf{P}_J w \mathbf{P}_{F(J)}$. Now, $\mathbf{Q}$ is F-stable if and only if $F(J) = J$ and $w \in W_J$. Given $I \subset S$ and $w \in W$, we have $\mathcal{L}^{-1}(\mathbf{P}_I w \mathbf{P}_{F(I)}) \neq \varnothing$ and this shows the equivalence of the two theorems.

Remark 2. The condition *"$W_I w$ is not contained in a proper F-stable standard parabolic subgroup of W"* is equivalent to *"$W_I w W_{F(I)}$ is not contained in a proper F-stable standard parabolic subgroup of W"*.

The rest of this Note is devoted to the proof of Theorem 2. We fix a subset I of S and an element w of W. We first recall two elementary facts. If $I \subset J$, let $\tau_{IJ} : \mathcal{P}_I \to \mathcal{P}_J$ be the morphism of varieties that sends $\mathbf{P} \in \mathcal{P}_I$ to the unique parabolic subgroup of type J containing $\mathbf{P}$. It is surjective. Moreover,

$$\tau_{IJ}\big(\mathbf{X}_I(w)\big) \subset \mathbf{X}_J(w) \tag{1}$$

and

$$\tau_{IJ}^{-1}\big(\mathbf{X}_J(w)\big) = \bigcup_{W_I x W_{F(I)} \subset W_J w W_{F(J)}} \mathbf{X}_I(x). \tag{2}$$

First step: the "only if" part. Assume that there exists a proper F-stable subset J of S such that $W_I w \subset W_J$. Then, by 1, we have $\tau_{IJ}(\mathbf{X}_I(w)) \subset \mathbf{X}_J(1) = \mathcal{P}_J^F$. Since $\mathbf{G}^F$ acts transitively on $\mathcal{P}_J^F$, we get $\tau_{IJ}(\mathbf{X}_I(w)) = \mathbf{X}_J(1)$. This shows that $\mathbf{X}_I(w)$ is not irreducible.

Second step: reduction to Borel subgroups. By the previous step, we can concentrate on the "if" part. So, from now on, we assume that $W_I w$ is not contained in a proper F-stable parabolic subgroup of W. Then, by 2, we have

$$\tau_{\varnothing I}^{-1}\big(\mathbf{X}_I(w)\big) = \bigcup_{x \in W_I w W_{F(I)}} \mathbf{X}(x).$$

Let v denote the longest element of $W_I w W_{F(I)}$. Then every element x of the double coset $W_I w W_{F(I)}$ satisfies $x \leqslant v$ (here, $\leqslant$ denotes the Bruhat order on W): this follows for instance from the fact that $\mathbf{P}_I w \mathbf{P}_{F(I)}$ is irreducible and is equal to $\bigcup_{x \in W_I w W_{F(I)}} \mathbf{B}w\mathbf{B}$. In particular, v is not contained in a proper F-stable parabolic subgroup of W.

Now, let $\mathbf{X}' = \bigcup_{x \in W_I w W_{F(I)}} \mathbf{X}(x)$. Note that $\overline{\mathbf{B}v\mathbf{B}} = \bigcup_{x \leqslant v} \mathbf{B}x\mathbf{B}$, hence $\overline{\mathcal{L}^{-1}(\mathbf{B}v\mathbf{B})} = \bigcup_{x \leqslant v} \mathcal{L}^{-1}(\mathbf{B}x\mathbf{B})$ since $\mathcal{L}$ is open. So, $\overline{\mathbf{X}(v)} = \bigcup_{x \leqslant v} \mathbf{X}(x)$ and we deduce that

$$\mathbf{X}(v) \subset \mathbf{X}' \subset \overline{\mathbf{X}(v)}.$$

So, since $\tau_{\varnothing I}(\mathbf{X}') = \mathbf{X}_I(w)$, it is enough to show that $\mathbf{X}(v)$ is irreducible. In other words, we may, and we will, assume that $I = \varnothing$.

Third step: smooth compactification. Let $(s_1, \ldots, s_n)$ be a finite sequence of elements of S. Let

$$\widehat{\mathbf{X}}(s_1, \ldots, s_n) = \left\{(\mathbf{B}_1, \ldots, \mathbf{B}_n) \in \mathcal{B}^n \mid \left(\mathbf{B}_n, F(\mathbf{B}_1)\right) \in \overline{\mathcal{O}(s_n)} \text{ and } (\mathbf{B}_i, \mathbf{B}_{i+1}) \in \overline{\mathcal{O}(s_i)} \text{ for } 1 \leqslant i \leqslant n-1\right\}.$$

If $\ell(s_1 \cdots s_n) = n$, then $\widehat{\mathbf{X}}(s_1, \ldots, s_n)$ is a smooth compactification of $\mathbf{X}(s_1 \cdots s_n)$ (see [1, Lemma 9.11]): in this case,

$$\mathbf{X}(s_1 \cdots s_n) \textit{ is irreducible if and only if } \widehat{\mathbf{X}}(s_1, \ldots, s_n) \textit{ is irreducible}. \tag{3}$$

Note that $(\mathbf{B}, \ldots, \mathbf{B}) \in \widehat{\mathbf{X}}(s_1, \ldots, s_n)$. We denote by $\widehat{\mathbf{X}}^\circ(s_1, \ldots, s_n)$ the connected (i.e. irreducible) component of $\widehat{\mathbf{X}}(s_1, \ldots, s_n)$ containing $(\mathbf{B}, \ldots, \mathbf{B})$. Let $H(s_1, \ldots, s_n) \subset \mathbf{G}^F$ be the stabilizer of $\widehat{\mathbf{X}}^\circ(s_1, \ldots, s_n)$. Let us now prove the following fact:

$$\textit{if } 1 \leqslant i_1 < \cdots < i_r \leqslant n, \textit{ then } H(s_{i_1}, \ldots, s_{i_r}) \subset H(s_1, \ldots, s_n). \tag{4}$$

Proof of (4). The map $f : \widehat{\mathbf{X}}(s_{i_1}, \ldots, s_{i_r}) \to \widehat{\mathbf{X}}(s_1, \ldots, s_n)$ defined by

$$f(\mathbf{B}_1, \ldots, \mathbf{B}_1) = \big(\mathbf{B}_1, \ldots, \underbrace{\mathbf{B}_1}_{\substack{i_1\text{-th} \\ \text{position}}}, \mathbf{B}_2, \ldots, \underbrace{\mathbf{B}_{r-1}}_{\substack{i_{r-1}\text{-th} \\ \text{position}}}, \mathbf{B}_r, \ldots, \underbrace{\mathbf{B}_r}_{\substack{i_r\text{-th} \\ \text{position}}}, F(\mathbf{B}_1), \ldots, F(\mathbf{B}_1)\big)$$

is a $\mathbf{G}^F$-equivariant morphism of varieties. Moreover,

$$f(\underbrace{\mathbf{B}, \ldots, \mathbf{B}}_{r \text{ times}}) = (\underbrace{\mathbf{B}, \ldots, \mathbf{B}}_{n \text{ times}}).$$

In particular, $f(\widehat{\mathbf{X}}^\circ(s_{i_1}, \ldots, s_{i_r}))$ is contained in $\widehat{\mathbf{X}}^\circ(s_1, \ldots, s_n)$. This proves the expected inclusion between stabilizers.

Last step: twisted Coxeter element. The quotient variety $\mathbf{G}^F \backslash \mathcal{L}^{-1}(\mathbf{B}w\mathbf{B}) \simeq \mathbf{B}w\mathbf{B}$ is irreducible, hence $\mathbf{G}^F \backslash \mathbf{X}(w)$ is irreducible as well. So,

$$\mathbf{G}^F \textit{ permutes transitively the irreducible components of } \mathbf{X}(w). \tag{5}$$

Let $w = s_1 \cdots s_n$ be a reduced decomposition of W as a product of elements of S. By (3) and (5), it suffices to show that $H(s_1, \ldots, s_n) = \mathbf{G}^F$. Since w does not belong to any F-stable proper parabolic subgroup of W, there exists a sequence $1 \leqslant i_1 < \cdots < i_r \leqslant n$ such that $(s_{i_k})_{1 \leqslant k \leqslant r}$ is a family of representatives of F-orbits in S. By (4), we have $H(s_{i_1}, \ldots, s_{i_r}) \subset H(s_1, \ldots, s_n)$. But, by [3, Proposition 4.8], $\mathbf{X}(s_{i_1}, \ldots, s_{i_r})$ is irreducible so, again by (3) and (5), $H(s_{i_1}, \ldots, s_{i_r}) = \mathbf{G}^F$. Therefore, $H(s_1, \ldots, s_n) = \mathbf{G}^F$, as expected.

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