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Probability Theory/Functional Analysis

The invertibility of adapted perturbations of identity on the Wiener space

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Abstract

Let (W, H, μ) be the classical Wiener space. Assume that $U = I_W + u$ is an adapted perturbation of identity, i.e., $u : W \to H$ is adapted to the canonical filtration of W. We give some sufficient analytic conditions on u which imply the invertibility of the map U. ***To cite this article: A.S. Üstünel, M. Zakai, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

L'inversibilité des perturbations d'identité adaptées sur l'espace de Wiener. Soit (W, H, μ) l'espace de Wiener. Soit $U = I_W + u$ une perturbation d'identité adaptée, i.e., $u : W \to H$ est adaptée à la filtration canonique de W. Nous donnons quelques conditions suffisantes qui impliquent l'inversibilité de l'application U. ***Pour citer cet article : A.S. Üstünel, M. Zakai, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Preliminaries

Let $W = C_0([0, 1])$ be the Banach space of continuous functions on $[0, 1]$, with its Borel sigma field denoted by $\mathcal{F}$. We denote by H the Cameron–Martin space, namely the space of absolutely continuous functions on $[0, 1]$ with square integrable Lebesgue density:

$$H = \left\{ h \in W \colon h(t) = \int_0^t \dot{h}(s)\,\mathrm{d}s,\ |h|_H^2 = \int_0^1 \left|\dot{h}(s)\right|^2 \mathrm{d}s < \infty \right\}.$$

μ denotes the classical Wiener measure on $(W, \mathcal{F})$, $(\mathcal{F}_t, t \in [0, 1])$ is the filtration generated by the paths of the Wiener process $(t, w) \to W_t(w)$, where $W_t(w)$ is defined as $w(t)$ for $w \in W$ and $t \in [0, 1]$. We shall recall briefly

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some well-known functional analytic tools on the Wiener space, we refer the reader to [4,3,5] or to [6] for further details: $(P_\tau, \tau \in \mathbb{R}_+)$ denotes the semi-group of Ornstein–Uhlenbeck on W, defined as

$$P_\tau f(w) = \int_W f\left(e^{-\tau} w + \sqrt{1 - e^{-2\tau}}\, y\right)\mu(dy).$$

Let us recall that $P_\tau = e^{-\tau \mathcal{L}}$, where $\mathcal{L}$ is the number operator. We denote by ∇ the Sobolev derivative which is the extension (with respect to the Wiener measure) of the Fréchet derivative in the Cameron–Martin space direction. The iterates of ∇ are defined similarly. Note that, if f is real valued, then ∇f is a vector and if u is an H-valued map, then ∇u is a Hilbert–Schmidt (on H) operator valued map whenever defined. If Z is a separable Hilbert space and if $p > 1, k \in \mathbb{R}$, we denote by $\mathbb{D}_{p,k}(Z)$ the μ-equivalence classes of Z-valued measurable mappings ξ, defined on W such that $(I + \mathcal{L})^{k/2}\xi$ belongs to $L^p(\mu, Z)$ and this set, equipped with the norm

$$\|\xi\|_{p,k} = \left\|(I + \mathcal{L})^{k/2}\xi\right\|_{L^p(\mu,Z)} \tag{1}$$

becomes a Banach space. From the Meyer inequalities, we know that the norm defined by

$$\sum_{k=0}^{n} \left\|\nabla^k \xi\right\|_{L^p(\mu, Z\otimes H^{\otimes k})}, \quad n \in \mathbb{N},$$

is equivalent to the norm $\|\xi\|_{p,n}$ defined by (1). We denote by δ the adjoint of ∇ under μ and recall that, whenever $u \in \mathbb{D}_{p,0}(H)$ for some $p > 1$ is adapted, then δu is equal to the Itô integral of the Lebesgue density of u:

$$\delta u = \int_0^1 \dot{u}_s \, dW_s.$$

2. A sufficient condition for invertibility

Assume that $u : W \to H$ is adapted, i.e., $u(t) = \int_0^t \dot{u}_s \, ds$, $t \in [0, 1]$ and that $\dot{u}_s$ is $\mathcal{F}_s$-measurable for almost all $s \in [0, 1]$. We suppose that $\rho(-\delta u)$ defined as

$$\rho(-\delta u) = \exp\left[-\delta u - \frac{1}{2}|u|_H^2\right]$$

is the terminal value of a uniformly integrable martingale. We shall assume that u is in $\mathbb{D}_{2,0}(H)$. We have

Theorem 1. *Assume that u satisfies the hypothesis above. For $\tau \in [0, 1]$, define u_τ as to be $P_\tau u$, where P_τ is the Ornstein–Uhlenbeck semigroup and assume also that $E[\rho(-\delta u_\tau)] = 1$ for $\tau \in [0, 1]$. Then the adapted perturbation of identity $U = I_W + u$ is invertible provided that*

$$E\left[\int_0^1 \left|(I_H + \nabla u_\tau)^{-1}\mathcal{L}u_\tau\right|_H \rho(-\delta u_\tau)\, d\tau\right] < \infty. \tag{2}$$

Proof. Note that the map u_τ is again adapted and $H - C^1$ (in fact it is even $H - C^\infty$, cf. [7]). This means that there exists a negligible set $N \subset W$ (in fact its capacity is null [6]) with $H + N \subset N$, such that, for any $w \in N^c$, the map $h \to u_\tau(w + h)$ is continuously Fréchet differentiable on H. Consequently $U_\tau = I_W + u_\tau$ satisfies the change of variables formula: for any $f \in C_b(W)$,

$$E\left[f \circ U_\tau \rho(-\delta u_\tau)\right] = E\left[f(w) N_\tau(w)\right],$$

where N_τ is the multiplicity function of U_τ, namely the cardinality of the set $U_\tau^{-1}(\{w\})$ (cf. [7]). Since $E[\rho(-\delta u_\tau)] = 1$, it follows that $N_\tau = 1$ μ-almost surely and this implies the existence of the inverse of U_τ which is denoted as V_τ. Note that V_τ is of the form $V_\tau = I_W + v_\tau$, where $v_\tau : W \to H$ and that the image of μ under V_τ, denoted as $V_\tau \mu$, is equivalent to μ with the Radon–Nikodym density

$$\frac{dV_\tau \mu}{d\mu} = \rho(-\delta u_\tau). \tag{3}$$

We also have $v_\tau = -u_\tau \circ V_\tau$. We shall prove that $\lim_{\tau\to 0} v_\tau$ exists in $L^0(\mu, H)$. Note that $\tau \to v_\tau$ is differentiable on $(0, 1)$ and we have

$$\frac{\mathrm{d}v_\tau}{\mathrm{d}\tau} = -\big((I_H + \nabla u_\tau)^{-1}\mathcal{L}u_\tau\big) \circ V_\tau. \tag{4}$$

Since

$$|v_\beta - v_\alpha| \leqslant \int_\alpha^\beta \left|\frac{\mathrm{d}v_\tau}{\mathrm{d}\tau}\right|_H \mathrm{d}\tau,$$

and since $L^0(\mu, H)$ is complete, in order to show that $\lim_{\alpha,\beta\to 0} \mu(\{|v_\alpha - v_\beta| > c\}) = 0$, for any $c > 0$, it suffices to show that

$$E\int_0^\kappa \left|\frac{\mathrm{d}v_\tau}{\mathrm{d}\tau}\right| \mathrm{d}\tau < \infty,$$

for some $\kappa > 0$. From the relations (3) and (4), we obtain

$$E\int_\alpha^\beta \left|\frac{\mathrm{d}v_\tau}{\mathrm{d}\tau}\right|_H \mathrm{d}\tau = E\int_\alpha^\beta \left|\big((I_H + \nabla u_\tau)^{-1}\mathcal{L}u_\tau\big) \circ V_\tau\right|_H \mathrm{d}\tau$$
$$= E\int_\alpha^\beta \left|(I_H + \nabla u_\tau)^{-1}\mathcal{L}u_\tau\right|_H \rho(-\delta u_\tau)\,\mathrm{d}\tau.$$

Hence the hypothesis (2) implies the existence of the limit $\lim_{\tau\to 0} v_\tau$ in $L^1(\mu, H)$ which we shall denote by v. Since $v_\tau = -u_\tau \circ V_\tau$ and since $(\rho(-\delta u_\tau), \tau \in [0, 1])$ is uniformly integrable, $V\mu$ is absolutely continuous with respect to μ and we have also the identity $v = -u \circ V$, where $V = I_W + v$. Now it is easy to see that $U \circ V = V \circ U = I_W$ μ-almost surely. □

Combining Theorem 1 with the inequality of T. Carleman (cf. [1] or [2], Corollary XI.6.28) which says:

$$\left\|\det\nolimits_2(I_H + A)(I_H + A)^{-1}\right\| \leqslant \exp\frac{1}{2}\big(\|A\|_2^2 + 1\big),$$

for any Hilbert–Schmidt operator A, where the left hand side is the operator norm, $\det_2(I_H + A)$ denotes the modified Carleman–Fredholm determinant and $\|\cdot\|_2$ denotes the Hilbert–Schmidt norm, we get

Theorem 2. *Assume that $u \in \mathbb{D}_{2,1}(H)$ such that $E[\rho(-\delta u_\tau)] = 1$ and that*

$$E\left[\mathrm{e}^{\frac{1}{2}\|\nabla u\|_2^2} \int_0^1 P_\tau\big(\rho(-\delta u_\tau)|\mathcal{L}u_\tau|_H\big)\,\mathrm{d}\tau\right] < \infty.$$

Then U satisfies the conclusions of Theorem 1.

Proof. The integrand in the relation (2) can be upperbounded as follows:

$$\left|(I_H + \nabla u_\tau)^{-1}\mathcal{L}u_\tau\right|_H \leqslant \exp\frac{1}{2}\big(\|\nabla u_\tau\|_2^2 + 1\big)|\mathcal{L}u_\tau|_H$$
$$\leqslant |\mathcal{L}u_\tau|_H P_\tau\left(\exp\frac{1}{2}\big(\|\nabla u\|_2^2 + 1\big)\right),$$

where the second line follows from the Jensen inequality. Here there is no term with $\det_2$ since, ∇u_τ being quasi-nilpotent, its Carleman–Fredholm determinant is always equal to one. We then use the symmetry of P_τ with respect to μ. □

Corollary 1. *Suppose that* u *is adapted,* $E[\rho(-\delta u_\tau)] = 1$ *for all* $\tau \in [0,1]$. *Let* $\varepsilon > 0$ *be given and assume further that* $u \in \mathbb{D}_{\frac{\varepsilon+1}{\varepsilon},2}(H)$ *and that the following relation holds*:

$$E\left[\left(1 + \mathrm{e}^{-\mathrm{e}(1+\varepsilon)\delta u}\right)\exp\left(\frac{1+\varepsilon}{2}\|\nabla u\|_2^2\right)\right] < \infty. \tag{5}$$

Then $U = I_W + u$ *is* μ*-almost surely invertible.*

Proof. Let C_ε represent the left-hand side of the relation (5), then using the Hölder inequality we get

$$E\left[\int_0^1 \left|(I_H + \nabla u_\tau)^{-1}\mathcal{L}u_\tau\right|_H \rho(-\delta u_\tau)\,\mathrm{d}\tau\right] \leqslant C_\varepsilon^{\frac{1}{1+\varepsilon}}\|u\|_{\frac{1+\varepsilon}{\varepsilon},2}.$$

Hence the conclusion follows. □

Remark. If we take $\varepsilon = 1$ in Corollary 1, then it is easy to see, using the Wiener chaos expansion for $E[|\mathcal{L}P_\tau u|_H^2]$ that

$$E\int_0^1 |\mathcal{L}P_\tau u|_H^2\,\mathrm{d}\tau \leqslant \|u\|_{2,1}^2.$$

Remark. In the case where u is not adapted, the condition (5) with $\varepsilon = 1$ is sufficient for the measure theoretic degree of the map U to be one as it is proven in Theorem 9.3.2 of [7].

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