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Partial Differential Equations

Multi-brackets of differential operators and compatibility of PDE systems

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Abstract

We establish an efficient compatibility criterion for an overdetermined system of generalized complete intersection type in terms of multi-brackets. ***To cite this article: B. Kruglikov, V. Lychagin, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Multi-crochets d'opérateurs différentiels et compatibilité des systèms d'EDP. Nous établissons un critère de compatibilité efficace pour un système détérminé de type intersection complète généralisée en termes de multi-crochets. ***Pour citer cet article : B. Kruglikov, V. Lychagin, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Soit π un fibré de dimension m sur une variété de dimension n, et soient $F_i \in C^\infty(J^{l_i}\pi)$, $i = 1, \dots, m+1$, des fonctions lisses sur les espaces de jets. On peut les assimiler à des opérateurs différentiels (non-linéaires) d'ordres l_i.

Nous définissons un multi-crochet $\{F_1, \dots, F_{m+1}\} \in C^\infty(J^l\pi)$ d'ordre $l = l_1 + \cdots + l_{m+1} - 1$. Quand π est le fibré trivial, si l'on dénote par $\ell_i(F)$ la i-ème coordonnées de la linéarisation $\ell(F)$, nous obtenons :

$$\{F_1, \dots, F_{m+1}\} = \frac{1}{m!} \sum_{\alpha \in \mathbf{S}_m, \beta \in \mathbf{S}_{m+1}} (-1)^\alpha (-1)^\beta \ell_{\alpha(1)}(F_{\beta(1)}) \circ \cdots \circ \ell_{\alpha(m)}(F_{\beta(m)})(F_{\beta(m+1)}).$$

Définition. Un système $\mathcal{E} = \langle F_1, \dots, F_r \rangle \subset J^k(\pi)$ de r équations différentielles $\bigcap\{F_i = 0 \mid 1 \leqslant i \leqslant r\}$ est une *intersection complète généralisée* si

1. $m < r \leqslant n + m - 1$;
2. La variété caractéristique projective complexe $\mathrm{Char}^{\mathbb{C}}_{x_k}(\mathcal{E}) \subset \mathbb{P}(T^*_x)^{\mathbb{C}}$ est de codimension $r - m + 1$ en chaque point $x_k \in \mathcal{E}$;
3. Le faisceau caractéristique sur $\mathrm{Char}^{\mathbb{C}}_{x_k}(\mathcal{E})$ admet des fibres de dimension 1.

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Les intersections complète, introduites dans [7], satisfont les propriétés ci-dessus et sont des systèmes particuliers de type Cohen–Macaulay considérés dans [7].

Soient $\mathcal{J}_s(F_1,\dots,F_r)=\langle\hat{\Delta}F_i\colon\ \Delta\in\mathrm{Diff}_{s-l_i}(\mathbf{1},\mathbf{1})|1\leqslant i\leqslant r\rangle\subset C^\infty(J^s\pi)$ les idéaux engendrés par $F_1,\dots,F_r$ et leurs dérivées totales.

Théorème. *Soit $\mathcal{E}=\langle F_1,\dots,F_r\rangle\subset J^k(\pi)$ un système d'équations différentielles.*

1. *Si $\mathcal{E}$ est formellement intégrable, alors tous les multi-crochets s'annulent pour le système* :
$$\{F_{i_1},\dots,F_{i_{m+1}}\}\in\mathcal{J}_{i_1+\cdots+i_{m+1}-1}(F_1,\dots,F_r),$$
pour tous $1\leqslant i_1<\cdots<i_{m+1}\leqslant r$.
2. *Si $\mathcal{E}$ est une intersection complète généralisée, alors $\mathcal{E}$ est formellement intégrable si et seulement si les multi-crochets s'annulent pour le système* :
$$\{F_{i_1},\dots,F_{i_{m+1}}\}\in\mathcal{J}_{i_1+\cdots+i_{m+1}-1}(F_1,\dots,F_r),$$
pour tous $1\leqslant i_1<\cdots<i_{m+1}\leqslant r$.

1. Determinants over non-commutative algebras

Let $\mathbf{k}$ be a commutative algebra over a field $\mathbb{F}$ of characteristic 0. Consider a monoidal category of $\mathbf{k}$-$\mathbf{k}$ bimodules and let $\mathbf{A}$ be an associative algebra in this category. In other words, let $\mathbf{A}$ be an associative $\mathbf{k}$-$\mathbf{k}$-algebra. For any left $\mathbf{k}$-module $\mathbf{V}$ we turn $\mathbf{A}\otimes_{\mathbb{F}}\Lambda^{\cdot}\mathbf{V}$, where $\Lambda^{\cdot}\mathbf{V}=\bigoplus_{i\geqslant 0}\Lambda^i\mathbf{V}$ is the exterior algebra of the module $\mathbf{V}$, into an associative $\mathbb{F}$-algebra by setting
$$(a\otimes_{\mathbb{F}}\alpha)\cdot(b\otimes_{\mathbb{F}}\beta)=ab\otimes_{\mathbb{F}}\alpha\wedge\beta,$$
where $a,b\in\mathbf{A}$, $\alpha,\beta\in\Lambda^{\cdot}\mathbf{V}$.

Assume that $\Lambda^m\mathbf{V}\simeq\mathbf{k}$ for some natural $m>0$, and let $\Omega\in\Lambda^m V$ be a basis m-vector. We define a determinant
$$\det=\det_\Omega:\mathbf{V}_{\mathbf{A}}^{\otimes_{\mathbb{F}}m}=\mathbf{V}_{\mathbf{A}}\otimes_{\mathbb{F}}\cdots\otimes_{\mathbb{F}}\mathbf{V}_{\mathbf{A}}\to\mathbf{A}$$
as
$$\xi_1\cdots\xi_m=\det(\xi_1,\dots,\xi_m)\Omega,$$
where $\mathbf{V}_{\mathbf{A}}=\mathbf{A}\otimes_{\mathbf{k}}\mathbf{V}$ is the left $\mathbf{A}$-module and $\xi_1,\dots,\xi_m\in\mathbf{V}_{\mathbf{A}}$.

In the case of free module $\mathbf{V}$ we come to a version of Cayley determinant:
$$\det(\xi_1,\dots,\xi_m)=\sum_{\sigma\in\mathbf{S}_m}(-1)^\sigma\xi_{1\sigma(1)}\cdots\xi_{m\sigma(m)},$$
where $\xi_{ij}\in\mathbf{A}$ is a j-coordinate of $\xi_i\in\mathbf{V}_{\mathbf{A}}$.

The permutation group $\mathbf{S}_m$ acts on $\mathbf{V}_{\mathbf{A}}^{\otimes m}$ in the natural way, and we define $\det_\sigma:\mathbf{V}_{\mathbf{A}}^{\otimes m}\to\mathbf{A}$ for any permutation $\sigma\in\mathbf{S}_m$ as
$$\det_\sigma(\xi_1,\dots,\xi_m)=\det(\xi_{\sigma^{-1}(1)},\dots,\xi_{\sigma^{-1}(m)}),$$
and the antisymmetric determinant as
$$\mathrm{Det}=\frac{1}{m!}\sum_{\sigma\in\mathbf{S}_m}\det_\sigma:\Lambda_{\mathbb{F}}^m(\mathbf{V}_{\mathbf{A}})\to\mathbf{A}.$$

Using this determinant we define the multi-bracket $\Lambda_{\mathbb{F}}^{m+1}(\mathbf{V}_{\mathbf{A}})\to\mathbf{V}_{\mathbf{A}}$ by the following formula:
$$\{\xi_1,\dots,\xi_{m+1}\}=\sum_{i=1}^{m+1}(-1)^{i-1}\,\mathrm{Det}(\xi_1,\dots,\hat{\xi}_i,\dots,\xi_{m+1})\xi_i.$$

2. Multi-brackets of differential operators

Here we apply the above construction to differential operators on a manifold M^n with $\mathbf{A} = \mathrm{Diff}(\mathbf{1},\mathbf{1})$ be the algebra of scalar linear differential operators ($\mathbf{1}$ is the trivial one-dimensional bundle over M) and $\mathbf{k} = C^\infty(M,\mathbb{R})$ (see, for example, [11,10,5]).

Let π be a smooth vector bundle. Then $\mathrm{Diff}(\pi,\mathbf{1}) = \mathbf{A}\otimes_{\mathbf{k}} C^\infty(\pi^*) = \mathbf{V_A}$ for $\mathbf{V} = C^\infty(\pi^*)$. Assume that π is orientable bundle and $\dim\pi = m$. Then $\Lambda^m(\pi^*)$ is a trivial 1-dimensional bundle, and picking a volume form $\Omega \in \Lambda^m\mathbf{V}$ we get the multi-bracket $\{\nabla_1,\dots,\nabla_{m+1}\} \in \mathrm{Diff}(\pi,\mathbf{1})$ for scalar-valued differential operators $\nabla_i \in \mathrm{Diff}(\pi,\mathbf{1})$ on π.

The order of the bracket (which is again a differential operator on π) is $l_1 + \cdots + l_{m+1} - 1$ if $\mathrm{ord}\,\nabla_i = l_i$.

Note that for the trivial 1-dimensional bundle $\pi = \mathbf{1}$ this bracket becomes the usual commutator of scalar differential operators: $\{\nabla_1,\nabla_2\} = \nabla_1\nabla_2 - \nabla_2\nabla_1$, $\nabla_i \in \mathbf{A}$.

To define a multi-bracket for non-linear scalar-valued differential operators F_i on vector bundle π we identify them with functions on the jet-spaces $J^k\pi$. The algebra $\mathbf{A}$ acts in the natural way on the algebra $C^\infty(J^\infty\pi)$ and convert $\mathbf{A}_\pi = C^\infty(J^\infty\pi)\otimes_{\mathbf{k}}\mathbf{A}$ into $\mathbf{k}$-$\mathbf{k}$ algebra. For any function $F \in C^\infty(J^k\pi)$, considered as a non-linear operator, the linearization $\ell(F)$ (see [5]) belongs to $\mathbf{A}_\pi\otimes_{\mathbf{k}}C^\infty(\pi^*)$, and therefore the bracket $\{\ell(F_1),\dots,\ell(F_{m+1})\} \in \mathbf{A}_\pi\otimes_{\mathbf{k}}C^\infty(\pi^*)$ is well defined if π is orientable and the volume form Ω is fixed. One can check that the bracket is a linearization of a function $\{F_1,\dots,F_{m+1}\} \in C^\infty(J^\infty\pi)$. In other words:

$$\ell\big(\{F_1,\dots,F_{m+1}\}\big) = \big\{\ell(F_1),\dots,\ell(F_{m+1})\big\}.$$

This defines the skew-symmetric multi-bracket:

$$C^\infty\big(J^{l_1}\pi\big)\otimes_{\mathbb{R}}\cdots\otimes_{\mathbb{R}} C^\infty\big(J^{l_{m+1}}\pi\big) \to C^\infty\big(J^l\pi\big),$$

where $l = l_1 + \cdots + l_{m+1} - 1$.

Remark that for $\pi = \mathbf{1}$ this bracket coincides with the Jacobi (or Lagrange for first order operators) brackets [9,5].

If π is a trivial vector bundle and $\ell(F) = (\ell_1(F),\dots,\ell_m(F))$, then

$$\{F_1,\dots,F_{m+1}\} = \frac{1}{m!}\sum_{\alpha\in\mathbf{S}_m,\,\beta\in\mathbf{S}_{m+1}} (-1)^\alpha(-1)^\beta \ell_{\alpha(1)}(F_{\beta(1)})\circ\cdots\circ\ell_{\alpha(m)}(F_{\beta(m)})(F_{\beta(m+1)}).$$

3. Main result

We consider overdetermined systems of differential equations $\mathcal{E} = \langle F_1,\dots,F_r\rangle$, which are defined by functions $F_i \in C^\infty(J^{l_i}\pi)$, $i = 1,\dots,r$, and $r > m$.

Define ideals $\mathcal{J}_s(F_1,\dots,F_r) = \langle\hat{\Delta}F_i\colon\ \Delta \in \mathrm{Diff}_{s-l_i}(\mathbf{1},\mathbf{1})\rangle \subset C^\infty(J^s\pi)$ generated by $F_1,\dots,F_r$ and their total derivatives.

Definition. We call a system $\mathcal{E} = \langle F_1,\dots,F_r\rangle \subset J^k(\pi)$ of r differential equations a *generalized complete intersection* if

1. $m < r \leqslant n + m - 1$, where $n = \dim M$;
2. The complex projective characteristic variety $\mathrm{Char}^{\mathbb{C}}_{x_k}(\mathcal{E}) \subset \mathbb{P}(T^*_x)^{\mathbb{C}}$ has codimension $r - m + 1$ at each point $x_k \in \mathcal{E}$.
3. The characteristic sheaf $\mathcal{K}$ over $\mathrm{Char}^{\mathbb{C}}_{x_k}(\mathcal{E})$ has fibers of dimension 1 everywhere (see [5,7] or the next section for definitions).

The complete intersections, introduced in [7], satisfy the above properties and they are particular systems of Cohen–Macaulay type considered in [7].

Let us also define the reduced multi-bracket by the formula

$$[F_{i_1},\dots,F_{i_{m+1}}]_{\mathcal{E}} = \{F_{i_1},\dots,F_{i_{m+1}}\} \bmod \mathcal{J}_{i_1+\cdots+i_{m+1}-1}(F_1,\dots,F_r).$$

Theorem. *Let $\mathcal{E} = \langle F_1, \dots, F_r\rangle \subset J^k(\pi)$ be a system of differential equations.*

1. *If $\mathcal{E}$ is formally integrable, then all multi-brackets vanish due to the system*:

$$[F_{i_1}, \dots, F_{i_{m+1}}]_{\mathcal{E}} = 0,$$

for all $1 \leqslant i_1 < \dots < i_{m+1} \leqslant r$.

2. *If $\mathcal{E}$ is a generalized complete intersection, then $\mathcal{E}$ is a formally integrable if and only if the multi-brackets vanish due to the system*:

$$[F_{i_1}, \dots, F_{i_{m+1}}]_{\mathcal{E}} = 0,$$

for all $1 \leqslant i_1 < \dots < i_{m+1} \leqslant r$.

It should be noted that in general the equation should be prolonged to sufficiently high order of jets to achieve formal integrability, but with our hypotheses (which are a kind of general position for overdetermined systems with specified range of r) we describe precisely to which order one should prolong and calculate obstractions to integrability.

In particular, we get the following compatibility criterion for scalar PDEs:

Corollary. *Let $\mathcal{E} = \langle F_1, \dots, F_r\rangle \subset J^k(\mathbf{1})$ be a system of complete intersection type. Then the system $\mathcal{E}$ is formally integrable if and only if all pair-wise Mayer brackets $[F_i, F_j]_{\mathcal{E}}$ vanish.*

This result, generalizing [6,7], was presented in [8] with an extra technical assumption. This additional condition can now be removed.

4. Sketch of the proof

We will consider for simplicity only the case of linear PDEs of the same order. Let Δ be a differential operator of order l acting from vector bundle π to vector bundle ν. Consider the corresponding equation $\mathcal{E}_l = \mathrm{Ker}(\Delta) \subset J^l(\pi)$ and its prolongations $\mathcal{E}_{k+l} \subset J^{k+l}(\pi)$. Under the conditions of the theorem the prolongations $\mathcal{E}_j$ exist up to the order $j \leqslant s = ml + l - 1$ and the only obstruction to integrability belongs to the Spencer δ-cohomology group $H^{s-1,2}(\mathcal{E})$. It can be identified with our collection of multi-brackets $[F_{i_1}, \dots, F_{i_{m+1}}]_{\mathcal{E}}$.

To see this we consider the **A**-module homomorphism

$$\phi^\Delta : \mathrm{Diff}(\nu, \mathbf{1}) \to \mathrm{Diff}(\pi, \mathbf{1}), \qquad \phi^\Delta(\nabla) = \nabla \circ \Delta$$

and let $\mathcal{E}^*$ be the cokernel, which is the inductive limit of

$$\mathrm{Diff}_k(\nu, \mathbf{1}) \xrightarrow{\phi_k^\Delta} \mathrm{Diff}_{k+l}(\pi, \mathbf{1}) \to \mathcal{E}^*_{k+l} \to 0.$$

System $\mathcal{E}$ is formally integrable if and only if the $C^\infty(M)$-modules $\mathcal{E}^*_i$ are projective and the natural maps $\pi^*_{i+1,i} : \mathcal{E}^*_i \to \mathcal{E}^*_{i+1}$ are injective.

We consider then the corresponding symbolic module

$$\mathcal{M}_\Delta = \mathrm{Coker}\, \sigma_\Delta,$$

where $\sigma_\Delta : \mathbf{S}T \otimes \nu^* \to \mathbf{S}T \otimes \pi^*$ is the dual symbol of the operator Δ.

Its annihilator is the *characteristic ideal* $I(\Delta)$ and the set of its zeros—the *characteristic variety* $\mathrm{Char}(\Delta)$. It can be characterized as follows [3,11,4].

For $p \in T^*_x \setminus 0$ let $\mathfrak{m}(p) \subset S(T_x) = \bigoplus_{i \geqslant 0} S^i T_x$ be the maximal ideal of homogeneous polynomials vanishing at p. Then localization $(\mathcal{M}_\Delta)_{\mathfrak{m}(p)} \neq 0$ if and only if the covector p is characteristic. The set of characteristic covectors p form the characteristic variety $\mathrm{Char}^{\mathbb{C}}_{x_k}(\mathcal{E}) \subset \mathbb{P}(T^*_x)^{\mathbb{C}}$ and the localizations $(\mathcal{M}_\Delta)_{\mathfrak{m}(p)} \neq 0$ for $p \in \mathrm{Char}^{\mathbb{C}}_{x_k}(\mathcal{E})$ form the *characteristic sheaf* $\mathcal{K}$ over it.

The condition of generalized complete intersection implies that the Fitting ideal of the symbolic module satisfies: $\mathrm{Fitt}_0(\mathcal{M}_\Delta) = \mathrm{Ann}\, \mathcal{M}_\Delta$ and the module $\mathbf{S}T / \mathrm{Ann}\, \mathcal{M}_\Delta$ is Cohen–Macaulay [1].

Then the Buchsbaum–Rim [2] complex $\mathcal{C}^1$:

$$0 \to S^{r-m-1}V^\star \otimes \Lambda^r U \xrightarrow{\partial} S^{r-m-2}V^\star \otimes \Lambda^{r-1}U \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Lambda^{m+1}U \xrightarrow{\varepsilon} U \xrightarrow{\varphi} V$$

gives a resolution of the module $\mathcal{M}_\Delta$.

Here $U = \mathbf{S}T \otimes \nu^*$, $V = \mathbf{S}T \otimes \pi^*$, $\star$ is the dualization over $\mathbf{S}T$, ∂ is the multiplication by the unit element $e \in V \otimes V^\star \subset SV \otimes \Lambda V^\star$ ($\Lambda V^\star$ acts on ΛU via the map $\Lambda\psi^\star$) and ε is a special splice map [2].

We obtain a differential syzygy for $\mathcal{E}^*$ from the symbolic one by taking our multi-bracket instead of ε. In order that $\mathcal{E}$ be formally integrable, this latter bracket should vanish due to the equation.

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