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Partial Differential Equations

# Some asymptotic properties for solutions of one-dimensional advection–diffusion equations with Cauchy data in $L^p(\mathbb{R})$

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**Abstract**

We state and discuss a number of fundamental asymptotic properties of solutions $u(\cdot,t)$ to one-dimensional advection–diffusion equations of the form $u_t + f(u)_x = (a(u)u_x)_x$, $x \in \mathbb{R}$, $t > 0$, assuming initial values $u(\cdot,0) = u_0 \in L^p(\mathbb{R})$ for some $1 \leqslant p < \infty$. ***To cite this article: P. Braz e Silva, P.R. Zingano, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***


**Résumé**

**Quelques propriétés asymptotiques des solutions des équations d'avection–diffusion unidimensionnelles aux données initiales dans $L^p(\mathbb{R})$.** Nous établissons plusieurs propriétés asymptotiques fondamentales des solutions $u(\cdot,t)$ des équations d'avection–diffusion du type $u_t + f(u)_x = (a(u)u_x)_x$, $x \in \mathbb{R}$, $t > 0$, aux données initiales dans l'espace de Lebesgue $L^p(\mathbb{R})$, où $1 \leqslant p < \infty$. ***Pour citer cet article : P. Braz e Silva, P.R. Zingano, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***


## 1. Introduction

We examine some properties of solutions $u(\cdot,t)$ to the Cauchy problem for scalar advection–diffusion equations

$$u_t + f(u)_x = \big(a(u)u_x\big)_x, \quad x \in \mathbb{R},\ t > 0, \tag{1a}$$

$$u(\cdot,0) = u_0 \in L^p(\mathbb{R}), \quad 1 \leqslant p < \infty, \tag{1b}$$

where $a(\cdot)$ and $f(\cdot)$ are given smooth functions. We assume that $a(\mathrm{u}) \geqslant \mu > 0$ for some fixed constant $\mu$ and all values of u concerned. Under these conditions, it is known that problem (1) admits a unique, smooth (classical), globally defined solution $u(\cdot,t) \in C^0([0,\infty[, L^p(\mathbb{R}))$, which is bounded for $t > 0$ and satisfies (1b) in $L^p$ sense, that is, $\|u(\cdot,t) - u_0\|_{L^p(\mathbb{R})} \to 0$ as $t \to 0$, see e.g. [2–4] and Section 2 below. Several additional properties of $u(\cdot,t)$ are given next.

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## 2. Decay estimates

We first state an important energy-type inequality for $u(\cdot,t)$.

**Theorem 2.1.** *Assume $a(\cdot), f(\cdot) \in C^1$ with $a(\cdot)$ bounded below by some constant $\mu > 0$. Then, for each $q \geqslant \max\{p, 2\}$, the solution $u(\cdot,t) \in C^0([0,\infty[, L^p(\mathbb{R}))$ of the Cauchy problem* (1) *satisfies*

$$T^{\frac{q}{2p}} \|u(\cdot,T)\|^q_{L^q(\mathbb{R})} + q(q-1)\mu \int_0^T t^{\frac{q}{2p}} \int_{\mathbb{R}} |u(x,t)|^{q-2} |u_x(x,t)|^2 \,dx\,dt$$
$$\leqslant 2\left(\frac{q}{2p}\right)^{\frac{1}{2}(\frac{q}{p}+1)} \left(1-\frac{1}{q}\right)^{-\frac{1}{2}(\frac{q}{p}-1)} \|u_0\|^q_{L^p(\mathbb{R})} \mu^{-\frac{1}{2}(\frac{q}{p}-1)} T^{\frac{1}{2}} \tag{2}$$

*for all $T > 0$.*

Theorem 2.1 can be proved adapting the method discussed in [5] to our present needs. Decay estimates for $u(\cdot,t)$ are readily obtained from inequality (2). Indeed, by the maximum principle and choosing $q = 4p$, inequality (2) yields the supnorm estimate

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leqslant C_p \|u_0\|_{L^p(\mathbb{R})} (\mu t)^{-\frac{1}{2p}}, \quad C_p := 4^{\frac{1}{p}} \left(4 - \frac{1}{p}\right)^{-\frac{1}{2p}} \tag{3}$$

for all $t > 0$. We note that the minimal value for $C_p$ is not known; its particular value given here is not optimal. Now, since $\|u(\cdot,t)\|_{L^p(\mathbb{R})} \leqslant \|u_0\|_{L^p(\mathbb{R})}$ for all $t \geqslant 0$, it follows from a simple interpolation estimate that

$$\|u(\cdot,t)\|_{L^r(\mathbb{R})} \leqslant C_p^{1-\frac{p}{r}} \|u_0\|_{L^p(\mathbb{R})} (\mu t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})}, \quad \forall t > 0 \tag{4}$$

for all $p \leqslant r \leqslant \infty$. Therefore, solutions decay in $L^r$ for any $r > p$. Using standard estimates for fundamental solutions of linear parabolic problems (see [1,2]), one obtains decay rates for the derivatives of $u(\cdot,t)$ as well, e.g.

$$\|u_x(\cdot,t)\|_{L^r(\mathbb{R})} \leqslant C(p,r,\mu,K_p,t_0) \|u_0\|_{L^p(\mathbb{R})} t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}}, \quad \forall t \geqslant t_0 \tag{5}$$

for each $t_0 > 0$. Similar bounds hold for the other derivatives. Here, the constant $C(p,r,\mu,K_p,t_0)$ depends on the particular functions $a(\cdot)$ and $f(\cdot)$, on the values of $p, r, \mu, t_0$, and on $K_p$, a bound for $\|u_0\|_{L^p(\mathbb{R})}$, i.e., $K_p > 0$ chosen such that

$$\|u_0\|_{L^p(\mathbb{R})} \leqslant K_p. \tag{6}$$

## 3. Asymptotic behavior, $p = 1$

For $t \gg 1$, more detailed behavior of $u(\cdot,t)$ can be obtained from Theorem 3.1 below. This theorem follows from the estimates given in Section 2. Here, we assume $a, f \in C^2$, with $f$ Hölder continuous at 0. We also assume $a(\mathsf{u}) \geqslant \mu > 0$ for all u, as before.

**Theorem 3.1.** *Let $u_0 \in L^1(\mathbb{R})$, and let $v(\cdot,t) \in C^0([0,\infty[, L^1(\mathbb{R}))$ be (any) solution of the Burgers equation*

$$v_t + f'(0)v_x + f''(0)vv_x = a(0)v_{xx}, \quad x \in \mathbb{R},\ t > 0 \tag{7}$$

*having the same mass as $u(\cdot,t)$, i.e., $\int_{\mathbb{R}} v(x,0)\,dx = \int_{\mathbb{R}} u_0(x)\,dx$. Then, one has*

$$\lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{1}{r})} \|u(\cdot,t) - v(\cdot,t)\|_{L^r(\mathbb{R})} = 0 \tag{8}$$

*for each $1 \leqslant r \leqslant \infty$, uniformly in $r$.*

Now, solutions of (7) can be studied in detail through explicit representation formulas obtained with the so-called Hopf–Cole transformation, see [6]. In many cases, these properties can be recast for (1a) using Theorem 3.1 above, as illustrated by the following two results.

**Theorem 3.2.** *Let $u(\cdot,t), \hat{u}(\cdot,t) \in C^0([0,\infty[, L^1(\mathbb{R}))$ be solutions of Eq.* (1a) *corresponding to initial states $u_0, \hat{u}_0 \in L^1(\mathbb{R})$, respectively, with the same mass. Then, one has*

$$\lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{1}{r})} \|u(\cdot,t) - \hat{u}(\cdot,t)\|_{L^r(\mathbb{R})} = 0, \tag{9}$$

*for all $1 \leqslant r \leqslant \infty$, uniformly in $r$.*

**Theorem 3.3.** *Let $u(\cdot,t) \in C^0([0,\infty[, L^1(\mathbb{R}))$ be the solution of problem* (1) *for some given initial state $u_0 \in L^1(\mathbb{R})$ with mass $m$. Then, for each $1 \leqslant r \leqslant \infty$,*

$$\lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{1}{r})} \|u(\cdot,t)\|_{L^r(\mathbb{R})} = \gamma_r(m),$$

*where, if $f''(0) \neq 0$, the quantity $\gamma_r(m)$ is given by*

$$\gamma_r(m) = \frac{|m|}{\sqrt{4\pi a(0)}} \left(4a(0)\right)^{\frac{1}{2r}} \frac{2a(0)}{mf''(0)} \left(1 - \mathrm{e}^{-\frac{mf''(0)}{2a(0)}}\right) \|\mathcal{F}\|_{L^r(\mathbb{R})},$$

*for $\mathcal{F} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ defined by*

$$\mathcal{F}(x) = 2\mathrm{e}^{-x^2} \Big/ \left(1 + \mathrm{e}^{-\frac{mf''(0)}{2a(0)}} - \left(1 - \mathrm{e}^{-\frac{mf''(0)}{2a(0)}}\right)\mathrm{erf}(x)\right).$$

*Here,* $\mathrm{erf}(x)$ *is the error function, and $\gamma_r(0) = 0$. If $f''(0) = 0$, $\gamma_r(m)$ is given by*

$$\gamma_r(m) = \frac{|m|}{\sqrt{4\pi a(0)}} \left(\frac{4\pi a(0)}{r}\right)^{\frac{1}{2r}}.$$

The case $r = 1$ is worth explicit mention:

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{L^1(\mathbb{R})} = |m| \quad \text{and} \quad \lim_{t\to\infty} \|u(\cdot,t) - \hat{u}(\cdot,t)\|_{L^1(\mathbb{R})} = 0$$

for any two solutions $u(\cdot,t)$, $\hat{u}(\cdot,t)$ of Eq. (1a) transporting the same mass $m$.

## 4. Asymptotic behavior, $p > 1$

Lastly, we consider solutions $u(\cdot,t)$ of problem (1) when $p > 1$. In this case, taking cut-off approximations $v_R = u_0 \cdot \chi_{[-R,R]} \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ of the given initial data $u_0 \in L^p(\mathbb{R})$, and using results for the case $p = 1$, one obtains the following.

**Theorem 4.1.** *Let $u(\cdot,t) \in C^0([0,\infty[, L^p(\mathbb{R}))$ be the solution of problem* (1) *corresponding to an initial state $u_0 \in L^p(\mathbb{R})$, $p > 1$. Then, one has*

$$\lim_{t\to\infty} t^{\frac{1}{2}(\frac{1}{p}-\frac{1}{r})} \|u(\cdot,t)\|_{L^r(\mathbb{R})} = 0 \tag{10}$$

*for each $p \leqslant r \leqslant \infty$, uniformly in $r$.*

As for the heat equation, the convergence to zero in (10) can be arbitrarily slow (for suitable $u_0 \in L^p(\mathbb{R})$ verifying (6), $K_p > 0$ fixed). Therefore, no rates better than (4) can be given in general.

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