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Lie Algebras

Lie algebras generated by 3-forms

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Abstract

Let U be a real vector space, B an inner product on U and $T \in \wedge U^*$ a 3-form. The 3-form T defines two natural maps, $[\cdot,\cdot]_U : \wedge^2 U \to U$ and $\sigma : U \to \wedge^2 U^* \cong \mathrm{so}(U, B)$ given by $[x, y]_U = 2B^\sharp(T(x, y, \cdot))$ and $\sigma(x) = T(x, \cdot, \cdot)$. We show that $[\cdot,\cdot]_U$ is a Lie bracket if and only if $g_T \equiv \mathrm{Im}(\sigma)$ is a Lie subalgebra of $\mathrm{so}(U, B)$. ***To cite this article: R.P. Rohr, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Algèbres de Lie engendrées par des 3-formes. Soit U un espace vectoriel réel, B un produit euclidien sur U et $T \in \wedge U^*$ une 3-forme. La 3-forme T permet de définir deux applications, $[\cdot,\cdot]_U : \wedge^2 U \to U$ et $\sigma : U \to \wedge^2 U^* \cong \mathrm{so}(U, B)$ telles que $[x, y]_U = 2B^\sharp(T(x, y, \cdot))$ et $\sigma(x) = T(x, \cdot, \cdot)$. On va démontrer que $[\cdot,\cdot]_U$ est un crochet de Lie si et seulement si $g_T \equiv \mathrm{Im}(\sigma)$ est une sous-algèbre de Lie de $\mathrm{so}(U, B)$. ***Pour citer cet article : R.P. Rohr, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Soit $\mathfrak{g}$ une algèbre de Lie quadratique et $(\cdot,\cdot)$ sa forme bilinéaire symétrique invariante. On définit la 3-forme de Cartan $T \in \wedge^3 \mathfrak{g}^*$ par $T(x, y, z) = (x, [y, z])$, $x, y, z \in \mathfrak{g}$ (voir [2]). Cette 3-forme définit une classe dans cohomologie de $\mathfrak{g}$. De plus si $\mathfrak{g}$ est simple elle engendre $H^3(\mathfrak{g})$. On peut changer de point de vue. Soit U un espace vectoriel réel (de dimension finie) muni d'une forme bilinéaire symétrique non dégénérée B et $T \in \wedge^3 U^*$ une 3-forme. On considère le crochet antisymétrique suivant,

$$[\cdot,\cdot]_U : U \wedge U \to U,$$
$$x \wedge y \mapsto 2B^\sharp\big(T(x, y, \cdot)\big),$$

avec $B^\sharp : U^* \to U$ l'isomorphisme induit par B. Si $[\cdot, \cdot]_U$ satisfait l'identité de Jacobi, alors : U devient une algèbre de Lie avec $[\cdot, \cdot]_U$ comme crochet ; B devient une forme invariante sur U ; T est une 3-forme de Cartan. Considérons l'homomorphisme d'espaces vectoriels suivant,

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$$\sigma : U \to \wedge^2 U^* \cong \mathrm{so}(U, B),$$
$$x \mapsto \sigma(x) \equiv T(x, \cdot, \cdot).$$

On note $\mathfrak{g}_T \equiv \mathrm{Im}(\sigma)$ l'image de cette homomorphisme. Cette définition est inspirée de la Définition 3.1 de [1]. Le résultat principal de cette Note est le théorème suivant :

Théorème 0.1. *Supposons que B est définie positive. Alors, $[\cdot,\cdot]_U$ est un crochet de Lie si et seulement si $\mathfrak{g}_T$ est une sous algèbre de Lie de* $\mathrm{so}(U, B)$. *De plus* :

(i) *σ est un homomorphisme d'algèbre de Lie, i.e.* $[\sigma(x), \sigma(y)] = \sigma([x, y]_U)$;
(ii) $\mathrm{Ker}(\sigma)$ *est un idéal abélien* ;
(iii) *T est une* 3*-forme de Cartan pour B* (*i.e.* $T(x, y, z) = B([x, y]_U, z)$).

La preuve de ce théorème dans le sens direct est un exercice facile (voir Remarque 1). Par contre, la preuve de la réciproque nécessite les étapes suivantes :

(i) On démontre que si deux éléments ($\sigma(x)$ et $\sigma(y)$) de $\mathfrak{g}_T$ commutent alors $[x, y]_U = 0$ (voir Lemme 2.1). De ce lemme on déduit :
  (a) l'algèbre de Lie $\mathfrak{g}_T$ est semi-simple (voir Lemme 2.2) ;
  (b) sa décomposition en somme directe d'algèbres de Lie simples est donnée par $\mathfrak{g}_T = \bigoplus_i \mathfrak{g}_{T_i}$ avec $T = \sum_i T_i$ où $T_i \in \wedge^3 U_i^*$ et $U = \bigoplus_i U_i \oplus \mathrm{Ker}(\sigma)$ est la décomposition orthogonale correspondante (voir Lemme 2.4).
(ii) On montre que $\sigma(x) \cdot y \equiv [\sigma(x), y] = [x, y]_U$ définit une structure $\mathfrak{g}_T$-module sur U (voir Section 2). Puis on démontre que U est homomorphe au module adjoint de $\mathfrak{g}_T$ et que $\sigma : U \to \mathfrak{g}_T$ est un homomorphisme de $\mathfrak{g}_T$-modules (voir Section 3).

1. Introduction

Let $\mathfrak{g}$ be a quadratic Lie algebra, with $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$ its invariant bilinear form. We can define a 3-form $T \in \wedge^3 \mathfrak{g}^*$ by $T(x, y, z) = ([x, y], z)$, $x, y, z \in \mathfrak{g}$. This 3-form, called a Cartan 3-form (see [2]), defines a class in the cohomology of $\mathfrak{g}$. In particular, for $\mathfrak{g}$ simple, T generates $H^3(\mathfrak{g})$. We can change the point of view. Let U be a real vector space (of finite dimension), equipped with a nondegenerate symmetric bilinear form B, and let $T \in \wedge^3 U^*$ be a 3-form. We define a skew-symmetric bracket

$$[\cdot, \cdot]_U : U \wedge U \to U, \qquad x \wedge y \mapsto 2B^\sharp\big(T(x, y, \cdot)\big), \tag{1}$$

where $B^\sharp : U^* \to U$ is the isomorphism induced by B. If $[\cdot, \cdot]_U$ is a Lie bracket (i.e. $[\cdot,\cdot]_U$ satisfies the Jacobi identity), then the bilinear form B is invariant with respect to the adjoint action, i.e. $\forall x, y, z \in U$, $B([x, y]_U, z) + B(y, [x, z]_U) = 0$ and T is the Cartan 3-form. Furthermore, we define the vector space homomorphism

$$\sigma : U \to \wedge^2 U^* \cong \mathrm{so}(U, B), \qquad x \mapsto \sigma(x) \equiv T(x, \cdot, \cdot). \tag{2}$$

We denote its image by $\mathfrak{g}_T \equiv \mathrm{Im}(\sigma) \subseteq \mathrm{so}(U, B)$. This definition is inspired by Definition 3.1 of [1]. The main result of this Note is the following theorem:

Theorem 1.1. *Assume that B is positive definite. Then, $[\cdot,\cdot]_U$ is a Lie bracket if and only if $\mathfrak{g}_T$ is a Lie subalgebra of* $\mathrm{so}(U, B)$. *Moreover, in this case we have*:

(i) *σ is a Lie algebra homomorphism, i.e.* $[\sigma(x), \sigma(y)] = \sigma([x, y]_U)$,
(ii) $\mathrm{Ker}(\sigma)$ *is an Abelian ideal,*
(iii) *T is the Cartan* 3*-form for B* (*i.e.* $T(x, y, z) = B([x, y]_U, z)$).

Remark 1. Assume that $[\cdot,\cdot]_U$ is a Lie bracket. Then, proving that $\mathfrak{g}_T$ is a Lie subalgebra of $\mathrm{so}(U,B)$ is an elementary exercise: let $Cl(U,B)$ be the Clifford algebra and denote $[\sigma(x),y]=\sigma(x)y-y\sigma(x)$ the commutator in $Cl(U,B)$ (here we use the Chevalley isomorphism $Cl(U,B)\cong\wedge U^*$, see Theorem II.1.6, p. 41 of [3]). We have $[x,y]_U=2B^\sharp T(x,y,\cdot)=[\sigma(x),y]$, which implies $\sigma(x)\cong ad_x$. Moreover, $ad_{[x,y]_U}=[ad_x,ad_y]$ implies that $g_T\subseteq\mathrm{so}(U,B)$ is a Lie subalgebra and $\sigma([x,y]_U)=[\sigma(x),\sigma(y)]$, i.e. σ is a Lie algebra homomorphism.

The above argument works for any signature of the bilinear form B. Our proof of the reciprocal, presented in Sections 2 and 3, depends on the positivity of B. We conjecture that the theorem above hold true if the bilinear form is only nondegenerate.

2. Properties of $\mathfrak{g}_T$

In this section we assume that $\mathfrak{g}_T$ is a Lie subalgebra of $\mathrm{so}(U,B)$ and that B is positive definite. Below we list some useful properties of the map σ and of the Lie subalgebra $\mathfrak{g}_T$.

Lemma 2.1. *Let $x,y\in U$ such that $[\sigma(x),\sigma(y)]=0$. Then, $[x,y]_U=0$.*

Proof. Let $\{e_i\}_{i=1..n}$ by an orthonormal basis of U. By Theorem 16 of [5] (page 293), we have $\forall x,y\in U$, $[\sigma(x),\sigma(y)]=2\sum_{i=1}^n(\iota(e_i)\iota(x)T)\wedge(\iota(e_i)\iota(y)T)$ (here ι is the left contraction operator). If $[\sigma(x),\sigma(y)]=0$, we have that $\sum_{i=1}^n(\iota(e_i)\iota(x)T)\wedge(\iota(e_i)\iota(y)T)=0$. In particular, we obtain

$$0=\iota(x)\iota(y)\sum_{i=1}^n\bigl(\iota(e_i)\iota(x)T\bigr)\wedge\bigl(\iota(e_i)\iota(y)T\bigr)=-\sum_{i=1}^n\bigl(T(y,x,e_i)\bigr)^2,$$

which implies $[x,y]_U=0$. □

Remark 2. From the above lemma we deduce that the Lie algebra $\mathfrak{g}_T$ is either trivial or nonabelian. Indeed, if $[\sigma(x),\sigma(y)]=0$ for all $x,y\in U$, then $T(x,y,\cdot)=0$ and $\sigma=0$.

Lemma 2.2. *The Lie algebra $\mathfrak{g}_T$ is semisimple.*

Proof. $\mathfrak{g}_T$ is a reductive Lie algebra, since it is a Lie subalgebra of $\mathrm{so}(U,B)$ and B is positive definite. Then, we have $\mathfrak{g}_T=Z(\mathfrak{g}_T)\oplus[\mathfrak{g}_T,\mathfrak{g}_T]$. If $\sigma(x)\in Z(\mathfrak{g}_T)$ (the center of $\mathfrak{g}_T$), Lemma 2.1 implies $\iota(x)\iota(y)T=0$, $\forall y\in U$ and $\sigma(x)=0$. □

We are interested in the decomposition of $\mathfrak{g}_T$ into a direct sum of simple Lie algebras. For that, we need the following definition.

Definition 2.3. The 3-form T is called reducible if there exists a nontrivial orthogonal decomposition $U=U_1\oplus U_2\oplus\mathrm{Ker}(\sigma)$, and nontrivial elements $T_i\in\wedge^3U_i^*$, $i=1,2$, such that $T=T_1+T_2$. Otherwise we say that T is irreducible.

Lemma 2.4. *The Lie algebra $\mathfrak{g}_T$ is simple if and only if the 3-form T is irreducible. Moreover, the decomposition of $\mathfrak{g}_T$ into simple Lie subalgebras is given by*

$$\mathfrak{g}_T=\bigoplus_i\mathfrak{g}_{T_i},$$

where $T_i\in\wedge^3U_i^*$ are the irreducible components of $T=\sum_i T_i$, and $U=\mathrm{Ker}(\sigma)\oplus\bigoplus_i U_i$ is the corresponding orthogonal decomposition.

Proof. The proof in the $\Rightarrow$) direction is obvious. We will give the proof in the $\Leftarrow$) direction. Assume that $\mathfrak{g}_T$ decomposes into a direct sum of Lie algebras $\mathfrak{g}_T=\mathfrak{g}_1\oplus\mathfrak{g}_2$. This decomposition defines a decomposition of U into a direct sum of vector spaces $U=U_1\oplus U_2\oplus\mathrm{Ker}(\sigma)$, where $U_i=\sigma^{-1}(\mathfrak{g}_i)\cap\mathrm{Ker}(\sigma)^\perp$, $i=1,2$. This decomposition together with Lemma 2.1 implies that $\forall x \in U_1$ and $\forall y \in U_2$, $T(x, y, \cdot) = 0$. Then, $T = T_1 + T_2$, with $T_i \equiv T|_{\wedge^3 U_i^*}$, $i = 1, 2$, and we have $\mathfrak{g}_i = \mathfrak{g}_{T_i}$, $i = 1, 2$. Moreover, $U_1 \perp U_2$. Indeed, we have that $B(B^\sharp(\sigma(x)), y \wedge z) = B(x, [y, z]_U)$, which implies $(\mathrm{Ker}(B^\sharp\sigma))^\perp = \mathrm{Im}([\cdot, \cdot]_U)$. Then, for each $x \in U_1$ there exists $w_x \in \wedge^2 U_1$ such that $B^\sharp(\iota(w_x)T) = x$ (here ι is the left contraction operator), and we have $\forall x \in U_1$ and $\forall y \in U_2$ that $\iota(y)\iota(w_x)T = B(x, y) = 0$, i.e. $U_1 \perp U_2$. □

The vector space U is a $\mathfrak{g}_T$-module, with $\mathfrak{g}_T$-action given by $\sigma(x) \cdot y \equiv [\sigma(x), y]$, where the bracket is the commutator in the Clifford algebra $Cl(U, B)$ and we identify $Cl(U, B)$ and $\wedge U^*$ with the Chevalley isomorphism.

Lemma 2.5. *The $\mathfrak{g}_T$-module U has the following properties*:

(i) $\sigma(x) \cdot y = [x, y]_U$, $\forall x, y \in U$,
(ii) *U is a faithful module,*
(iii) $[\sigma(x), \sigma(y)] = 0 \Rightarrow \sigma(x) \cdot y = 0$,
(iv) $\sigma(x) \cdot y = -\sigma(y) \cdot x$, $\forall x, y \in U$,
(v) *U is irreducible if and only if $\mathfrak{g}_T$ is simple.*

Proof. The point (i) follows from the fact that in the Clifford algebra $Cl(U, B)$ we have $[x, \bullet] = 2\iota(x)\bullet$ (see Theorem 5, page 284 of [5]). The points (ii) and (iv) follow from the point (i), the point (iii) follows from the point (i) and Lemma 2.1 and the point (v) follows from the point (i) and Lemma 2.4. □

3. The homomorphism theorem

In this section, we prove a theorem that allows us to complete the proof of Theorem 1.1.

Theorem 3.1. *Let $\mathfrak{g}$ be a simple Lie algebra ($\mathfrak{h}$ is its Cartan subalgebra), U a faithful $\mathfrak{g}$-module, and $\sigma : U \to \mathfrak{g}$ a vector space isomorphism. Assume that the following properties hold true*:

P1 $\forall v, w \in U$, $\sigma(v) \cdot w = -\sigma(w) \cdot v$,
P2 $\forall v, w \in \sigma^{-1}(\mathfrak{h})$, $\sigma(v) \cdot w = 0$.

Then σ is an isomorphism of $\mathfrak{g}$-modules, i.e. $[\sigma(x), \sigma(y)] = \sigma(\sigma(x) \cdot y)$, $\forall x, y \in U$, and then $x \wedge y \to \sigma(x) \cdot y$ is a Lie bracket on U.

Proof. First we complexify $\mathfrak{g}$ and U. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha$ be the root space decomposition, $\mathcal{R}$ be the set of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$. This decomposition induces a decomposition of $U^{\mathbb{C}}$ by σ, i.e. $U^{\mathbb{C}} = U^0 \oplus \bigoplus_{\alpha \in \mathcal{R}} U^\alpha$, where $U^0 = \sigma^{-1}(\mathfrak{h}^{\mathbb{C}})$ and $U^\alpha = \sigma^{-1}(\mathfrak{g}_\alpha)$.

The structure of the $\mathfrak{g}^{\mathbb{C}}$-module $U^{\mathbb{C}}$ induces another decomposition, $U^{\mathbb{C}} = \bigoplus_{\lambda \in \mathcal{P}} U_\lambda$, where $\mathcal{P}$ is the set of weights of U and $U_\lambda \subseteq U$ is the subspace of weight λ. The subspace of weight zero is denoted by $U_0 \equiv U_{\lambda=0}$.

(i) The property P2 implies $U^0 \subseteq U_0$.
(ii) We prove that every root of $\mathfrak{g}^{\mathbb{C}}$ is a weight of U, i.e. $\mathcal{R} \subseteq \mathcal{P}$. Let $h \in U^0$ and $v^\alpha \in U^\alpha$. Then, property P1, the point (i) above and Lemma 20.1, page 107 of [4] imply that $\sigma(h) \cdot v^\alpha \in U_\alpha$. We shall show that for every root α there exists an element $h_\alpha \in U^0$ such as $\sigma(h_\alpha) \cdot v^\alpha \neq 0$. Indeed, if it is not the case, then there is $\beta \neq \alpha$ such that for some $v^\beta \in U^\beta$, $\sigma(v^\beta) \cdot v^\alpha \neq 0$ (because the module U is faithful). We have then $\forall h \in U^0$,

$$\underbrace{\sigma(h) \cdot \sigma(v^\beta) \cdot v^\alpha}_{=\beta(h)\sigma(v^\beta)\cdot v^\alpha} = -\sigma(h) \cdot \sigma(v^\alpha) \cdot v^\beta = -[\sigma(h), \sigma(v^\alpha)] \cdot v^\beta - \sigma(v^\alpha) \cdot (\sigma(h) \cdot v^\beta)$$

$$= \alpha(h) \underbrace{\sigma(v^\beta) \cdot v^\alpha}_{\text{of weight } \beta} + \underbrace{\sigma(v^\alpha) \cdot (\sigma(v^\beta) \cdot h)}_{\text{of weight } \alpha+\beta}.$$

This is a contradiction. Indeed if the last term vanishes, then $\alpha = \beta$, and if it is not the case then a nonvanishing eigenvector of weight $\alpha + \beta$ is proportional to an eigenvector of weight β.

(iii) The point (i) says that $U^0 \subseteq U_0$ and the point (ii) says that $\mathcal{R} \subseteq \mathcal{P}$. Then, for dimensional reasons, $U^0 = U_0$ and $\mathcal{R} = \mathcal{P}$. This implies that the $\mathfrak{g}$-module U is isomorphic to the adjoint module, i.e. there exists an isomorphism of $\mathfrak{g}$-modules $\Phi : \mathfrak{g} \to U$ such that $\Phi([\sigma(x), \sigma(y)]) = \sigma(x) \cdot \Phi(\sigma(y))$, $\Phi(\sigma(U^0)) = U_0$ and $\Phi(\sigma(U^\alpha)) = U_\alpha$. Moreover, the skew-symmetric map $x \wedge y \to \sigma(x) \cdot \Phi(\sigma(y))$ defines a Lie bracket on U. In the end of the proof we will show that $\Phi = \lambda\sigma^{-1}$, where $\lambda \in \mathbb{C}^*$.

(iv) We will prove that $U_\alpha = U^\alpha$ for all roots α. Let α be a root. Then, $\forall h \in U^0$ we have $\alpha(h) \neq 0$ iff $\sigma(h) \cdot v^\alpha \neq 0$. Indeed, from the point (ii) we know that there exist $h_\alpha \in U^0$ such that $\sigma(h_\alpha) \cdot v^\alpha \neq 0$. We have $\forall h \in U^0$ that

$$\alpha(h)\sigma(h_\alpha) \cdot v^\alpha = \sigma(h) \cdot \sigma(h_\alpha) \cdot v^\alpha = \alpha(h_\alpha)\sigma(h) \cdot v^\alpha.$$

We know that there exist a $h' \in U^0$ such that $\alpha(h') \neq 0$. This implies that $\alpha(h_\alpha) \neq 0$. If $\alpha(h) \neq 0$ then $\sigma(h) \cdot v^\alpha \neq 0$, and if $\sigma(h) \cdot v^\alpha \neq 0$ then $\alpha(h) \neq 0$. Moreover, from these equations we deduce that there exist a $x_\alpha \in U_\alpha$ such that for all $h \in U^0$ we have

$$\sigma(h) \cdot v^\alpha = \alpha(h)x_\alpha.$$

From this identity and because $\sigma(h) \cdot x_\alpha = \alpha(h)x_\alpha$, we deduce that there exist $k_\alpha \in U^0$ such that $v^\alpha = x_\alpha + k_\alpha$. We shall prove that $k_\alpha = 0$. Indeed, for all roots α and β we have $\sigma(v^\alpha) \cdot v^\beta = -\sigma(v^\beta) \cdot v^\alpha$, which implies

$$\underbrace{\sigma\big(v^\alpha\big) \cdot x_\beta + \sigma\big(v^\beta\big) \cdot x_\alpha}_{\text{of weight } \alpha+\beta} = \underbrace{\alpha(k_\beta)x_\alpha}_{\text{of weight } \alpha} + \underbrace{\beta(k_\alpha)x_\beta}_{\text{of weight } \beta}.$$

Then $\beta(k_\alpha) = 0$ for every root β, which implies $k_\alpha = 0$, i.e. $v^\alpha = x_\alpha$. Then $U_\alpha = U^\alpha$, and for all $h \in U_0$, $v^\alpha \in U^\alpha$ we have $[\sigma(h), \sigma(v^\alpha)] = \sigma(\sigma(h) \cdot v^\alpha)$.

(v) Let $v^\alpha \in U^\alpha$, $v^\beta \in U^\beta$ and $v^{\alpha+\beta} \in U^{\alpha+\beta}$ such that $[\sigma(v^\alpha), \sigma(v^\alpha)] = \sigma(v^{\alpha+\beta})$. We shall show that $\sigma(v^\alpha) \cdot v^\beta = v^{\alpha+\beta}$. Indeed, for all $h \in U_0$, we have

$$\underbrace{\sigma\big(v^{\alpha+\beta}\big) \cdot h}_{-(\alpha+\beta)(h)v^{\alpha+\beta}} = \underbrace{\sigma\big(v^\alpha\big) \cdot \sigma\big(v^\beta\big) \cdot h}_{-\beta(h)\sigma(v^\alpha)\cdot v^\beta} - \underbrace{\sigma\big(v^\beta\big) \cdot \sigma\big(v^\alpha\big) \cdot h}_{-\alpha(h)\sigma(v^\alpha)\cdot v^\beta}.$$

Then $\sigma(v^\alpha) \cdot v^\beta = v^{\alpha+\beta}$, i.e. $[\sigma(v^\alpha), \sigma(v^\alpha)] = \sigma(\sigma(v^\alpha) \cdot v^\beta)$.

(vi) From the point (iii) we know that $U_0 = U^0$, then $\forall h, h' \in U_0$, $[\sigma(h), \sigma(h')] = \sigma(\sigma(h) \cdot h')$. From the point (iv) we have $\forall \alpha \in \mathcal{R}$, $\forall v^\alpha \in U^\alpha$ and $\forall h \in U_0$ that $[\sigma(h), \sigma(v_\alpha)] = \sigma(\sigma(h) \cdot v_\alpha)$. And from the point (v) we have $\forall \alpha, \beta \in \mathcal{R}$, $\forall v^\alpha \in U^\alpha$ and $\forall v^\beta \in U^\beta$ that $[\sigma(v^\alpha), \sigma(v^\alpha)] = \sigma(\sigma(v^\alpha) \cdot v^\beta)$. We can then conclude $\forall x, y \in U$ that $[\sigma(x), \sigma(y)] = \sigma(\sigma(x) \cdot y)$, i.e. $\Phi = \lambda\sigma^{-1}$, $\lambda \in \mathbb{C}^*$. □

Now we can finalize the prove of Theorem 1.1. First we decompose the Lie algebra $\mathfrak{g}_T$ into a direct sum of simple Lie algebras, $\mathfrak{g}_T = \bigoplus_i \mathfrak{g}_{T_i}$, where $T_i \in \wedge^3 U_i^*$ are the irreducible components of T (see Lemma 2.4). Second we apply the Theorem 3.1 to each $\mathfrak{g}_{T_i}$. Finally we conclude that $[\cdot,\cdot]_U$ is a Lie bracket, σ is a Lie algebra homomorphism and with Lemma 2.1 that $\mathrm{Ker}(\sigma)$ is an Abelian ideal.

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