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Local strong solution for the incompressible Korteweg model ☆

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Abstract

We consider a Navier–Stokes type system with a Korteweg stress tensor, coupled with a concentration equation without diffusion. We give a result concerning the local in time existence (and uniqueness) of strong solution for any data (and the global in time existence defined in a interval $(0,T)$ for $T<+\infty$ fixed if data are small enough), in the case of a bounded domain $\Omega\subset\mathbb{R}^3$. ***To cite this article: M. Sy et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Solution local fort pour le modèle de Korteweg incompressible. On considère un système de type Navier–Stokes avec un tenseur de Korteweg, couplé avec une équation de concentration sans diffusion. On donne un premier résultat d'existence local en temps (et unicité) de solution forte pour des données quelconques (et d'existence globale en temps defini dans un interval $(0,T)$ pour $T<+\infty$ fixé si les données sont petites), dans le cas d'un domaine borné $\Omega\subset\mathbb{R}^3$. ***Pour citer cet article : M. Sy et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

On s'intéresse à l'étude de la régularité de solution d'un modèle de type Navier–Stokes avec un tenseur des contraintes spécifiques, introduit par Korteweg en 1901. Un tel modèle propose que la distribution de densité non uniforme (de concentration ou de température) induit des efforts et de la convection dans le fluide. Dans [4], il étudient le cas de deux liquides miscibles, dans lequel le tenseur est une tension interfaciale effective qui se relaxe avec le temps dû à l'effet de la diffusion. Ici, on considère un modèle similaire mais sans diffusion. Ainsi, il est plus difficile d'obtenir la régularité forte du modèle.

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Concrètement, le modèle qu'on étudie est dans un domaine $\Omega \subset \mathbb{R}^3$, suffisamment régulier, dont les inconnues sont la vitesse $\mathbf{u}(t,\mathbf{x})$, la pression $\pi(t,\mathbf{x})$, et la concentration $\rho(t,\mathbf{x})$. Le modèle est donné par (KM) (voir après). On prouve le résultat suivant :

Théorème 0.1. *Si $\rho_0 \in W^{2,p}(\Omega)$ et $\mathbf{u}_0 \in \mathbf{W}^{1,p}(\Omega)$, il existe $T_* \in (0,T]$ suffisamment petit ou $T_* = T$ et des données petites, tel qu'il existe une (seule) solution de (KM) dans $(0,T_*)$ avec $\rho \in L^\infty(0,T_*;W^{2,p}(\Omega))$, $\mathbf{u} \in L^p(0,T_*;\mathbf{W}^{2,p}(\Omega)) \cap L^\infty(0,T_*;\mathbf{W}^{1,p}(\Omega))$ et $\partial_t \mathbf{u} \in L^p(0,T_*;\mathbf{L}^p(\Omega))$.*

Les techniques pour l'obtention de solution régulière du modèle de type Navier–Stokes à densité variable ne peuvent pas s'appliquer dans ce cas car le couplage est fortement non linéaire, dû au terme $\Delta\rho\nabla\rho$, dans le système pour $\mathbf{u}$. Pour le traiter d'une façon convenable, on a utilisé des estimations de type non-hilbertien [5,7] et des techniques de point fixe.

1. Introduction

In this Note, we are interested in the study of regularity for a Navier–Stokes type model with a specific stress tensor, introduced for the first time by Korteweg, [3]. He proposed in 1901 that a nonuniform density (concentration or temperature) distribution could induce stresses and convection in a fluid. In [4], the authors study the case of two miscible liquids brought in contact, this tensor acts like an effective interfacial tension that relaxes with time due to the mass diffusion. We take a similar model that does not consider diffusion for the concentration ρ. Thus, it becomes more difficult to obtain the strong regularity. Such work is the aim of this paper. The unknowns are the solenoidal velocity $\mathbf{u}(t,\mathbf{x})$, the pressure $\pi(t,\mathbf{x})$, and the concentration $\rho(t,\mathbf{x})$. Moreover, we suppose that the fluid is confined in a three-dimensional smooth enough domain Ω. Denoting $Q = (0,T)\times\Omega$, $\Sigma = (0,T)\times\partial\Omega$, the model can be written as:

$$\text{(KM)}\quad \begin{cases} \partial_t\mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\pi = -k\nabla\rho\Delta\rho & \text{in Q},\\ \partial_t\rho + \mathbf{u}\cdot\nabla\rho = 0 \quad \nabla\cdot\mathbf{u} = 0 & \text{in } Q,\\ \mathbf{u} = \mathbf{0} \quad \text{on } \Sigma, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \rho|_{t=0} = \rho_0 & \text{in } \Omega. \end{cases}$$

As for the coefficients, $\nu > 0$ is the fluid viscosity, $\lambda > 0$ is the elasticity constant and $\gamma > 0$ is a relaxation in time constant. On the other hand, ρ_0 and $\mathbf{u}_0$ are the initial data. Note that the coupling term depending on ρ that appears in the system for $\mathbf{u}$ can be rewritten as $k\rho\nabla(\Delta\rho)$, using the following manipulation:

$$\rho\nabla\Delta\rho = \nabla(\rho\Delta\rho) - \nabla\cdot(\nabla\rho\otimes\nabla\rho) + \frac{1}{2}\nabla\big(|\nabla\rho|^2\big) = \nabla(\rho\Delta\rho) - \nabla\rho\Delta\rho,$$

and including the term $\nabla(\rho\Delta\rho)$ in the pressure term. Moreover, a conservative formulation of this term is $-k\nabla\cdot(\nabla\rho\otimes\nabla\rho)$, including $\nabla(\frac{1}{2}|\nabla\rho|^2)$ in the pressure term.

1.1. Functional spaces and equivalent norms

We denote by $\mathbf{H}$ and $\mathbf{V}$ the space of type L^2 and H^1 respectively, associated to the incompressibility and Dirichlet homogeneous boundary conditions for the velocity:

$$\mathbf{H} = \big\{\mathbf{u}\in\mathbf{L}^2(\Omega):\ \nabla\cdot\mathbf{u} = 0, \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0\big\}, \qquad \mathbf{V} = \big\{\mathbf{u}\in\mathbf{H}^1(\Omega):\ \nabla\cdot\mathbf{u} = 0, \mathbf{u}|_{\partial\Omega} = 0\big\}.$$

Moreover, we will do the estimates in $\mathbf{W}^{1,p}(\Omega)\cap\mathbf{V}$ or $\mathbf{W}^{2,p}(\Omega)\cap\mathbf{V}$, and we will take $\|\nabla\mathbf{v}\|_{L^p(\Omega)}$ and $\|\Delta\mathbf{v}\|_{L^p(\Omega)}$ as the equivalent norms in those spaces. In order to study the concentration of the fluid ρ, we consider the space:

$$W_N^{k,p}(\Omega) = \left\{\rho\in W^{k,p}(\Omega)\ \Big|\ \int_\Omega \rho(t)\,\mathrm{d}\Omega = \int_\Omega \rho_0\,\mathrm{d}\Omega,\ \forall t\right\} \equiv \rho_0 + W_{N,0}^{k,p}(\Omega) \quad (k=1,2,\ p\in[1,\infty))$$

where $W_{N,0}^{k,p}(\Omega) = \{\rho\in W^{k,p}(\Omega)\mid \int_\Omega\rho(t)\,\mathrm{d}\Omega = 0, \forall t\}$. Thus, $\|\nabla\rho\|_{L^p(\Omega)}$ and $\|\Delta\rho\|_{L^p(\Omega)}$ define norms in $W_N^{1,p}(\Omega)$ and $W_N^{2,p}(\Omega)$, respectively. Observe that the bold face notation is used for the vectorial spaces in order to make different from the scalar ones.

1.2. Previous results

Notice that under (weak) regularity hypothesis for the data $\mathbf{u}_0 \in \mathbf{H}$ and $\rho_0 \in H^1(\Omega)$, this model has global in time weak estimates, i.e. any weak solution (if exists) verifies the following estimates:

$$\mathbf{u} \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V}), \qquad \rho \in L^\infty\big(0,T;H^1(\Omega)\big). \tag{1}$$

Observe that those weak estimates are obtained taking the gradient of the equation of ρ and multiplying by $k\nabla\rho$; taking $\mathbf{u}$ as a test function for the equation of $\mathbf{u}$, and adding both expressions. That reads:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(k\|\nabla\rho\|^2_{L^2(\Omega)} + \|\mathbf{u}\|^2_{L^2(\Omega)}\big) + \nu\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} = 0.$$

However, these estimates are not regular enough in order to provide the convergence (in any sense) for the nonlinear term $\nabla\rho\Delta\rho$ (even for their conservative formulation $\nabla\cdot(\nabla\rho\otimes\nabla\rho)$).

2. Uniqueness criterium

Let $(\rho_1,\mathbf{u}_1,\pi_1)$ and $(\rho_2,\mathbf{u}_2,\pi_2)$ be two solutions of (KM). If we call $\bar\rho = \rho_1-\rho_2$, $\bar\pi = \pi_1-\pi_2$ and $\bar{\mathbf{u}} = \mathbf{u}_1-\mathbf{u}_2$, we can write the system verified by $(\bar\rho,\bar{\mathbf{u}},\bar\pi)$ as follows:

$$(\mathrm{KM}_u)\quad \begin{cases} \partial_t\bar\rho + (\mathbf{u}_1\cdot\nabla)\bar\rho + (\bar{\mathbf{u}}\cdot\nabla)\rho_2 = 0 & \text{in } Q,\\ \partial_t\bar{\mathbf{u}} - \nu\Delta\bar{\mathbf{u}} + (\mathbf{u}_2\cdot\nabla)\bar{\mathbf{u}} + (\bar{\mathbf{u}}\cdot\nabla)\mathbf{u}_1 + \nabla\bar\pi + k\Delta\rho_1\nabla\bar\rho + k\Delta\bar\rho\nabla\rho_2 = \mathbf{0} & \text{in } Q,\\ \nabla\cdot\bar{\mathbf{u}} = 0 \quad \text{in } Q, \quad \bar{\mathbf{u}} = \mathbf{0} \quad \text{on } \Sigma, \quad \bar{\mathbf{u}}|_{t=0} = \mathbf{0}, \quad \bar\rho|_{t=0} = 0 & \text{in } \Omega. \end{cases}$$

Thus, taking gradient in $(\mathrm{KM}_u)_1$, multiplying by $-k\nabla\bar\rho$ and multiplying $(\mathrm{KM}_u)_2$ by $\bar{\mathbf{u}}$, we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(k\|\nabla\bar\rho\|^2_{L^2(\Omega)} + \|\bar{\mathbf{u}}\|^2_{L^2(\Omega)}\big) + \nu\|\nabla\bar{\mathbf{u}}\|^2_{L^2(\Omega)} = -\int_\Omega (\bar{\mathbf{u}}\cdot\nabla)\mathbf{u}_1\cdot\bar{\mathbf{u}}\,\mathrm{d}\Omega - k\int_\Omega (\bar{\mathbf{u}}\cdot\nabla)\bar\rho\Delta\rho_1\,\mathrm{d}\Omega.$$

The right-hand side terms can be bounded as follows:

$$\int_\Omega (\bar{\mathbf{u}}\cdot\nabla)\mathbf{u}_1\cdot\bar{\mathbf{u}}\,\mathrm{d}\Omega \leqslant \|\bar{\mathbf{u}}\|^2_{L^4(\Omega)}\|\nabla\mathbf{u}_1\|_{L^2(\Omega)} \leqslant \bar\varepsilon\|\nabla\bar{\mathbf{u}}\|^2_{L^2(\Omega)} + C_{\bar\varepsilon}\|\nabla\mathbf{u}_1\|^4_{L^2(\Omega)}\|\bar{\mathbf{u}}\|^2_{L^2(\Omega)},$$

$$k\int_\Omega (\bar{\mathbf{u}}\cdot\nabla)\bar\rho\Delta\rho_1\,\mathrm{d}\Omega \leqslant k\|\bar{\mathbf{u}}\|_{L^6(\Omega)}\|\nabla\bar\rho\|_{L^2(\Omega)}\|\Delta\rho_1\|_{L^3(\Omega)} \leqslant \bar\varepsilon\|\nabla\bar{\mathbf{u}}\|^2_{L^2(\Omega)} + C_{\bar\varepsilon}\|\Delta\rho_1\|^2_{L^3(\Omega)}\|\nabla\bar\rho\|^2_{L^2(\Omega)}.$$

Therefore, if we want to conclude the uniqueness of weak solution by means of a Gronwall's Lemma, we need to impose that $\nabla\mathbf{u}_1 \in L^4(0,T;\mathbf{L}^2(\Omega))$ and $\rho_1 \in L^2(0,T;W^{2,3}(\Omega))$.

3. Stronger estimates

We want to obtain strong estimates for $\mathbf{u}$, π and ρ. Using the non-Hilbertian estimates for the Stokes problem [7], we find that for any $p>1$:

$$\|\partial_t\mathbf{u}\|^p_{L^p_t(Q)} + \|\mathbf{u}\|^p_{L^p_t(W^{2,p}(\Omega))} + \|\pi\|^p_{L^p_t(W^{1,p}(\Omega))} \leqslant C\big(\big\|(\mathbf{u}\cdot\nabla)\mathbf{u}\big\|^p_{L^p_t(Q)} + \|\Delta\rho\nabla\rho\|^p_{L^p_t(Q)}\big). \tag{2}$$

We bound the last term of (2) as $\|\Delta\rho\nabla\rho\|^p_{L^p(\Omega)} \leqslant \|\nabla\rho\|^p_{L^\infty(\Omega)}\|\Delta\rho\|^p_{L^p(\Omega)} \leqslant C\|\Delta\rho\|^{2p}_{L^p(\Omega)}$, for any $p>3$. In order to bound $\|\Delta\rho\|^p_{L^p(\Omega)}$, we take the Laplacian of $(\mathrm{KM})_2$ and multiply by $|\Delta\rho|^{p-2}\Delta\rho$, obtaining:

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\rho\|^p_{L^p(\Omega)} \leqslant{}& \int_\Omega |\Delta\mathbf{u}||\nabla\rho||\Delta\rho|^{p-1}\,\mathrm{d}\Omega + \int_\Omega |\nabla\mathbf{u}|\big|D^2\rho\big||\Delta\rho|^{p-1}\,\mathrm{d}\Omega\\ &+ \int_\Omega (\mathbf{u}\cdot\nabla)\Delta\rho|\Delta\rho|^{p-2}\Delta\rho\,\mathrm{d}\Omega := I_1 + I_2 + I_3(=0) = I_1 + I_2.\end{aligned}$$

We bound the I_i-terms as follows (recall that $p>3$):

$$I_1 \leqslant \|\Delta\mathbf{u}\|_{L^p(\Omega)}\|\nabla\rho\|_{L^\infty(\Omega)}\|\Delta\rho\|_{L^p(\Omega)}^{p-1} \leqslant C\|\Delta\mathbf{u}\|_{L^p(\Omega)}\|\Delta\rho\|_{L^p(\Omega)}^p,$$
$$I_2 \leqslant C\|\nabla\mathbf{u}\|_{L^\infty(\Omega)}\left\|D^2\rho\right\|_{L^p(\Omega)}^p \leqslant C\|\mathbf{u}\|_{W^{2,p}(\Omega)}\|\Delta\rho\|_{L^p(\Omega)}^p$$

which led us to the expression $\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\rho(t)\|_{L^p(\Omega)}^p \leqslant C\|\mathbf{u}\|_{W^{2,p}(\Omega)}\|\Delta\rho(t)\|_{L^p(\Omega)}^p$.

Thus, integrating directly we obtain:

$$\|\Delta\rho\|_{L_t^\infty(L^p(\Omega))}^p \leqslant \|\Delta\rho_0\|_{L^p(\Omega)}^p \mathrm{e}^{C\|\mathbf{u}\|_{L_t^1(W^{2,p})}}, \quad \forall t > 0. \tag{3}$$

Therefore, integrating in time the estimates for $\|\nabla\rho\Delta\rho\|_{L^p(\Omega)}^p$ and using (3), we obtain:

$$\int_0^t \|\Delta\rho\nabla\rho\|_{L^p(\Omega)}^p(s)\,\mathrm{d}s \leqslant Ct\|\Delta\rho\|_{L_t^\infty L^p(\Omega)}^{2p} \leqslant Ct\|\rho_0\|_{W^{2,p}(\Omega)}^{2p}\mathrm{e}^{2C\|\mathbf{u}\|_{L_t^1(W^{2,p}(\Omega))}},$$

On the other hand, the nonlinear term in **u** can be estimated as:

$$\int_0^t \left\|(\mathbf{u}\cdot\nabla)\mathbf{u}\right\|_{L^p(\Omega)}^p(s)\,\mathrm{d}s \leqslant \int_0^t \left\|\mathbf{u}(s)\right\|_{L^\infty(\Omega)}^p\left\|\nabla\mathbf{u}(s)\right\|_{L^p(\Omega)}^p\mathrm{d}s \leqslant C\int_0^t \left\|\mathbf{u}(s)\right\|_{W^{1,p}(\Omega)}^{2p}\mathrm{d}s \leqslant Ct\|\mathbf{u}\|_{L_t^\infty(W^{1,p}(\Omega))}^{2p}.$$

Returning to inequality (2), it becomes the following inequality:

$$\|\partial_t\mathbf{u}\|_{L_t^p(L^p(\Omega))}^p + \|\mathbf{u}\|_{L_t^p(W^{2,p}(\Omega))}^p + \|\pi\|_{L_t^p(W^{1,p}(\Omega))}^p$$
$$\leqslant Ct\|\mathbf{u}\|_{L_t^\infty(W^{1,p}(\Omega))}^{2p} + Ct\|\rho_0\|_{W^{2,p}(\Omega)}^{2p}\mathrm{e}^{2C\|\mathbf{u}\|_{L_t^1(W^{2,p}(\Omega))}}. \tag{4}$$

Observe that it is possible to show that (see Appendix A):

$$\|\mathbf{u}\|_{L_t^\infty(W^{1,p}(\Omega))}^p \leqslant \|\mathbf{u}_0\|_{W^{1,p}(\Omega)}^p + C\left(\|\partial_t\mathbf{u}\|_{L_t^p(L^p(\Omega))}^p + \|\mathbf{u}\|_{L_t^p(W^{2,p}(\Omega))}^p\right). \tag{5}$$

Thus, the inequality (4) can be rewritten in the following form:

$$\|\mathbf{u}\|_{L_t^\infty(W^{1,p}(\Omega))}^p + \|\partial_t\mathbf{u}\|_{L_t^p(L^p(\Omega))}^p + \|\mathbf{u}\|_{L_t^p(W^{2,p}(\Omega))}^p$$
$$\leqslant \|\mathbf{u}_0\|_{W^{1,p}(\Omega)}^p + Ct\left\{\|\mathbf{u}\|_{L_t^\infty(W^{1,p}(\Omega))}^{2p} + \|\rho_0\|_{W^{2,p}(\Omega)}^{2p}\exp\left(2C\|\mathbf{u}\|_{L_t^1(W^{2,p}(\Omega))}\right)\right\}. \tag{6}$$

4. The main result

Theorem 4.1. *If $\rho_0 \in W^{2,p}(\Omega)$ and $\mathbf{u}_0 \in \mathbf{W}^{1,p}(\Omega)$, there exist a time $T_* \in (0,T]$ either small enough or $T_* = T$ and small enough data, such that there is a (unique) solution of (KM) in $(0,T_*)$ with $\rho \in L^\infty(0,T_*;W^{2,p}(\Omega))$, $\mathbf{u} \in L^p(0,T_*;\mathbf{W}^{2,p}(\Omega)) \cap L^\infty(0,T_*;\mathbf{W}^{1,p}(\Omega))$ and $\partial_t\mathbf{u} \in L^p(0,T_*;\mathbf{L}^p(\Omega))$.*

Proof. We define the Banach space

$$\mathbf{X}(T) = \left\{\mathbf{u} \mid \mathbf{u} \in L^p\left(0,T;\mathbf{W}^{2,p}(\Omega)\right) \cap L^\infty\left(0,T;\mathbf{W}^{1,p}(\Omega)\right), \partial_t\mathbf{u} \in L^p\left(0,T;\mathbf{L}^p(\Omega)\right)\right\}$$

endowed with the norm

$$\|\mathbf{u}\|_{X(T)} = \left(\|\mathbf{u}\|_{L_T^p(W^{2,p})}^p + \|\mathbf{u}\|_{L_T^\infty(W^{1,p})}^p + \|\partial_t\mathbf{u}\|_{L_T^p(L^p)}^p\right)^{1/p}.$$

The argument we use is the following: Given a closed set A_r, $A_r = \{\mathbf{u}\colon \|\mathbf{u}\|_{X(T)}^p \leqslant r\} \subset \mathbf{X}(T)$ ($r > 0$ will be chosen), where we define an operator $R: A \subset \mathbf{E} \to \mathbf{E}$ as $\bar{\mathbf{u}} \in A \mapsto R(\bar{\mathbf{u}}) = \mathbf{u}$ being $\mathbf{E} = L^2(0,T;\mathbf{H}^1(\Omega))$ and **u** the solution of the following linear problem (of Stokes type):

$$\partial_t\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = -(\bar{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}} - k\nabla\rho\Delta\rho, \quad \nabla\cdot\mathbf{u} = \mathbf{0} \quad \text{in } Q, \quad \mathbf{u}|_\Sigma = \mathbf{0}, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega, \tag{7}$$

where the concentration ρ appearing on the right-hand side of $(7)_1$ is the solution of the transport problem:

$$\partial_t\rho + (\bar{\mathbf{u}}\cdot\nabla)\rho = 0 \quad \text{in } Q, \qquad \rho|_{t=0} = \rho_0 \quad \text{in } \Omega. \tag{8}$$

Then, a regular (and unique) solution of the problem is a fixed point of operator R. It suffices, in order to apply the Schauder's fixed point theorem, to verify the following statements:

1) *A_r is a convex compact set of* $\mathbf{E}$ (*for each* $r > 0$): Observe that A_r is a convex closed subspace of $\mathbf{X}(T)$. The compact results of Aubin–Lions type [6] let us to insert A_r compactly in $\mathbf{E}$.

2) *There exists* $r > 0$ *such that* $R(A_r) \subset A_r$: Suppose that $\bar{\mathbf{u}} \in A_r$. The main idea is to find $r > 0$ such that the associated convex set A_r verifies $R(A_r) \subset A_r$. Using (6), we get:

$$\|\mathbf{u}\|^p_{X(T)} \leqslant \|\mathbf{u}_0\|^p_{W^{1,p}(\Omega)} + CT\left\{r^2 + \|\rho_0\|^{2p}_{W^{2,p}(\Omega)} \exp\left(2CT^{(p-1)/p}r\right)\right\}.$$

Therefore, we chose either a small time T_* or small data in order to obtain $\|\mathbf{u}\|^p_{X(T_*)} \leqslant r$. Concretely,

– *for small time*: Let ρ_0 and $\mathbf{u}_0$ given. Taking $r > 2\|\mathbf{u}_0\|^p_{W^{1,p}(\Omega)}$, we chose T_* small enough in order to obtain the following inequality:

$$CT_*\left\{r^2 + \|\rho_0\|^{2p}_{W^{2,p}(\Omega)} \exp\left(2CT_*^{(p-1)/p}r\right)\right\} \leqslant r/2.$$

– *for small data*: Now, the time T is fixed. Then, we choose r small enough such that $CTr^2 \leqslant r/2$, and $\|\mathbf{u}_0\|_{W^{1,p}(\Omega)}$, $\|\rho_0\|_{W^{2,p}(\Omega)}$ small enough verifying:

$$\|\mathbf{u}_0\|^p_{W^{1,p}(\Omega)} + CT\|\rho_0\|^{2p}_{W^{2,p}(\Omega)} \exp\left(2CT^{(p-1)/p}r\right) \leqslant r/2.$$

3) *R is continuous from* $\mathbf{E}$ *into* $\mathbf{E}$: It suffices to prove sequential continuity, i.e., if $\bar{\mathbf{u}}_n \to \bar{\mathbf{u}}$ in $\mathbf{E}$, then $R(\bar{\mathbf{u}}_n) \to R(\bar{\mathbf{u}})$ in $\mathbf{E}$. Let $(\bar{\mathbf{u}}_n) \subseteq A_r$ such that $\bar{\mathbf{u}}_n \to \bar{\mathbf{u}}$ in $\mathbf{E}$. Since A_r is a closed bounded set of $\mathbf{X}(T)$ (and $\mathbf{X}(T)$ is a reflexive Banach space), $\bar{\mathbf{u}} \in A_r$ and $\bar{\mathbf{u}}_n \rightharpoonup \bar{\mathbf{u}}$ in $\mathbf{X}(T)$.

We consider $\mathbf{u}_n = R(\bar{\mathbf{u}}_n)$ and $\mathbf{u} = R(\bar{\mathbf{u}})$. In particular, $(\mathbf{u}_n) \subseteq A_r$, $\mathbf{u} \in A_r$ and there exists a subsequence $(\mathbf{u}_{n_k}) \subset (\mathbf{u}_n)$ and a limit function $\Upsilon \in A_r$ such that:

$$\mathbf{u}_{n_k} \to \Upsilon \quad \text{strongly in } \mathbf{E} \text{ and weakly in } \mathbf{X}(T). \tag{9}$$

It suffices to prove that $\Upsilon = \mathbf{u}$, being $\mathbf{u}$ the unique solution of system (7), because in this case the whole sequence $(\mathbf{u}_n)$ converges to $\mathbf{u}$. To this aim, we need to consider ρ_n the solution of (8) associated to the velocity $\bar{\mathbf{u}}_n$. Then, using (3), one has that ρ_n is bounded in $L^\infty(0,T;W^{2,p}(\Omega))$ (and $\partial_t\rho_n$ is bounded in $L^p(0,T;W^{1,p}(\Omega))$), hence (ρ_n) is relatively compact in $C([0,T];W^{1,p}(\Omega))$. Therefore, there exists a subsequence $(\rho_{n_k}) \subset (\rho_n)$ such that:

$$\rho_{n_k} \to \chi \quad \text{strongly in } C\left([0,T];W^{1,p}(\Omega)\right) \text{ and weakly-}\star \text{ in } L^\infty\left(0,T;W^{2,p}(\Omega)\right). \tag{10}$$

Then, taking the limit in the equation $\partial_t\rho_{n_k} + (\bar{\mathbf{u}}_{n_k}\cdot\nabla)\rho_{n_k} = 0$ in $(0,T)\times\Omega$, we obtain that $\chi = \rho$, where ρ is the solution of (8) for the velocity $\bar{\mathbf{u}}$. Since this solution is unique, the previous convergence holds for the whole sequence (ρ_n). The next step is to take the limit in the system:

$$\partial_t\mathbf{u}_n - \nu\Delta\mathbf{u}_n + \nabla\pi_n = -(\bar{\mathbf{u}}_n\cdot\nabla)\bar{\mathbf{u}}_n - k\nabla\rho_n\Delta\rho_n, \qquad \nabla\cdot\mathbf{u}_n = \mathbf{0} \quad \text{in } (0,T)\times\Omega. \tag{11}$$

Then, thanks to the convergences for $\bar{\mathbf{u}}_n$, (10) for the whole sequence (ρ_n) and $\chi = \rho$ and (11), one has that Υ is a solution of (7). Since the solution of system (7) is unique, we conclude that $\Upsilon = \mathbf{u}$ and the convergence (9) holds for the whole sequence $(\mathbf{u}_n)$. □

Appendix A. Proof of inequality (5)

Observe that $\|\nabla\mathbf{u}\|_{L^p(\Omega)}$ is a norm equivalent to $\|\mathbf{u}\|_{W^{1,p}(\Omega)}$ in $\mathbf{W}^{1,p}(\Omega)\cap\mathbf{V}$, and that: $\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{u}\|^p_{L^p(\Omega)} = \int_\Omega \nabla(\partial_t\mathbf{u})\cdot(|\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u})\,\mathrm{d}\Omega$. Thus, integrating by parts on the right-hand side:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{u}\|^p_{L^p(\Omega)} &\leqslant C\int_\Omega |\nabla\mathbf{u}|^{p-2}\left|D^2\mathbf{u}\right||\partial_t\mathbf{u}|\,\mathrm{d}\Omega \leqslant C\|\nabla\mathbf{u}\|^{p-2}_{L^p(\Omega)}\left\|D^2\mathbf{u}\right\|_{L^p(\Omega)}\|\partial_t\mathbf{u}\|_{L^p(\Omega)}\\ &\leqslant C\|\mathbf{u}\|^{p-1}_{W^{2,p}(\Omega)}\|\partial_t\mathbf{u}\|_{L^p(\Omega)} \leqslant C\left[\|\mathbf{u}\|^p_{W^{2,p}(\Omega)} + \|\partial_t\mathbf{u}\|^p_{L^p(\Omega)}\right].\end{aligned}$$

Therefore, for each $t > 0$, $\|\mathbf{u}\|^p_{L^\infty_t(W^{1,p}(\Omega))} \leqslant \|\mathbf{u}_0\|^p_{W^{1,p}(\Omega)} + C[\|\partial_t\mathbf{u}\|^p_{L^p_t(L^p(\Omega))} + \|\mathbf{u}\|^p_{L^p_t(W^{2,p}(\Omega))}]$.

Remark 1. It is possible to make similar estimates in a different space of strong solutions (see [1] for the case of an Oldroyd model), obtaining the following regularity for $s \in (1, +\infty)$ and $r \in (3, +\infty)$:

$$\mathbf{u} \in L^s\big(0, T; \mathbf{W}^{2,r}(\Omega)\big), \quad \partial_t \mathbf{u} \in L^s\big(0, T; \mathbf{L}^r(\Omega)\big), \quad \rho \in L^\infty\big(0, T; W^{2,r}(\Omega)\big), \quad \partial_t \rho \in L^s\big(0, T; W^{1,r}(\Omega)\big).$$

In this case, we use Giga–Sohr's results [2] instead of those of Ladyzhenskaya and Solonnikov [5].

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