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Dynamical Systems/Ordinary Differential Equations

Extension of invariant manifolds and applications

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Abstract

We state an extension theorem for invariant manifolds of diffeomorphisms near a 'normally hyperbolic' invariant torus. We apply this result in particular to the resolution of equations $\mathcal{L}_{U_j}(f)=\theta_j$ $(1\leqslant j\leqslant n-1)$ where the U_j's are linear diagonal vector fields and the θ_j's are germs at 0 of smooth functions on $\mathbf{R}^n$. ***To cite this article: B. Abbaci, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Prolongement de variétés invariantes et applications. Nous énonçons un théorème de prolongement de variété invariante et nous en donnons une application à la résolution des équations $\mathcal{L}_{U_j}(f)=\theta_j$ $(1\leqslant j\leqslant n-1)$ où les U_j sont des champs de vecteurs linéaires diagonaux et les θ_j des germes en 0 de fonctions C^∞ de $\mathbf{R}^n$ vérifiant certaines conditions. ***Pour citer cet article : B. Abbaci, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Hypothèses

Soit $h:(M,\Sigma)\to(M,\Sigma)$ un germe en $\Sigma:=\mathbf{T}^r\times\{(0,0,0)\}$ de difféomorphisme C^∞ de la variété $M:=\mathbf{T}^r\times\mathbf{R}^\ell\times\mathbf{R}^m\times\mathbf{R}^q$ donné par $h(\theta,x,y,z)=(e(\theta,x,y),f_+(\theta,x,y),f_-(\theta,x,y),g(\theta,x,y,z))$. On suppose que les germes en Σ de $W^+:=\mathbf{T}^r\times\mathbf{R}^\ell\times\{(0,0)\}$ et $W^-:=\mathbf{T}^r\times\{0\}\times\mathbf{R}^m\times\{0\}$ sont h-invariants et que celui de $W:=\mathbf{T}^r\times\mathbf{R}^\ell\times\mathbf{R}^m\times\{0\}$ a un contact infini avec son image par h le long de $W^+\cup W^-$. On suppose en outre (« hyperbolicité normale ») que les automorphismes linéaires $A_\theta\in\mathrm{GL}(\mathbf{R}^\ell)$ et $B_\theta\in\mathrm{GL}(\mathbf{R}^m)$ définis par $A_\theta:=\frac{\partial f_+}{\partial x}(\theta,0,0)$ et $B_\theta:=\frac{\partial f_-}{\partial y}(\theta,0,0)$ vérifient

$$\sup_\theta|A_\theta|<1\quad\text{et}\quad\sup_\theta\left|B_\theta^{-1}\right|<1. \tag{HN}$$

On a alors le théorème de prolongement suivant [1] :

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Théorème 0.1. *Sous ces hypothèses, tout germe en Σ de sous-variété V de $M \setminus W^-$ de classe C^∞, h-invariant et ayant un contact infini avec W le long de W^+ se prolonge de manière unique en un germe en Σ de sous-variété $\widetilde{V}$ de M de classe C^∞ invariant par h, qui a un contact infini avec W le long de W^-.*

Indications sur la démonstration [1]

Dans un voisinage de $\{y=0\}$ dans $M \setminus W^-$, la sous-variété V est le graphe d'une application $\phi : \mathbf{T}^r \times \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}^q$ de classe C^∞ telle que $\frac{\partial^j \phi}{\partial y^j}(\theta, x, 0) = 0$ pour tout j dans $\mathbf{N}$. On prouve le théorème en montrant que, dans un voisinage de $\{(x,y)=(0,0)\}$ *dans M*, la sous-variété V se prolonge de manière unique en le graphe invariant par h d'une application C^∞ (encore notée ϕ), et que $\frac{\partial^j \phi}{\partial x^j}(\theta, 0, y) = 0$ pour tout j dans $\mathbf{N}$.

Remarque 1. L'adhérence du domaine de définition de l'application ϕ initiale ne contient *a priori* aucun ouvert de W^-. L'unicité vient donc de l'invariance par le difféomorphisme h.

La démonstration du Théorème 0.1, qui figurera dans une publication plus développée [2], ressemble à celle du théorème qui fait l'objet de la section (A.6.1) de l'appendice 6 de [3]. Celui-ci, qui joue un rôle central dans la linéarisation des germes d'actions de $\mathbf{R}^p$ ([3], Chapitre 3), résulte immédiatement du Théorème 0.1, ainsi que les énoncés des sections (A.6.3) et (A.6.5) de l'appendice 6 de [3].

Plutôt que d'y revenir, nous allons prouver un énoncé [1] qui ne figure pas dans [3].

Une application du théorème de prolongement

Étant donné un entier $n \geqslant 2$, on pose $p := n-1$ et l'on considère p champs de vecteurs $U_1, \ldots, U_p$ linéaires diagonaux sur $\mathbf{R}^n$; ils commutent donc deux à deux. Si l'on note $U : \mathbf{R}^p \to \mathrm{gl}_n(\mathbf{R})$ l'application linéaire $v \mapsto \sum_{i=1}^p v_i U_i$, on a $U(v) = \sum_{i=1}^n \lambda_i(v) x_i \frac{\partial}{\partial x_i}$. On suppose que les formes linéaires $\lambda_i : \mathbf{R}^p \to \mathbf{R}$ ainsi définies vérifient les hypothèses suivantes :

(H1) L'action U est *hyperbolique*, c'est-à-dire que les λ_i sont p à p indépendantes sur $\mathbf{R}$.
(H2) Elle est dans le *domaine de Siegel*, c'est-à-dire que l'enveloppe convexe des λ_i contient l'origine.

Étant donnés des germes $\theta_j : (\mathbf{R}^n, 0) \to \mathbf{R}$ de fonctions C^∞, on veut résoudre les équations

$$\mathcal{L}_{U_j} f = \theta_j \quad \text{pour } 1 \leqslant j \leqslant p. \tag{E_j}$$

Théorème 0.2. *Si les θ_j satisfont aux conditions $\mathcal{L}_{U_j}\theta_i = \mathcal{L}_{U_i}\theta_j$ pour $1 \leqslant i < j \leqslant p$ (évidemment nécessaires) et sont nuls à tous les ordres le long des hyperplans de coordonnées, alors les germes de fonctions $f : (\mathbf{R}^n, 0) \to \mathbf{R}$ de classe C^∞, nuls à tous les ordres le long des hyperplans de coordonnées et vérifiant (E_j) pour $1 \leqslant j \leqslant p$ forment un espace affine de dimension infinie.*

Remarque 2. On en déduit le même théorème pour des germes U_j nuls en 0 de champs de vecteurs commutant deux à deux et « génériques », car ils sont simultanément C^∞-linéarisables [3].

Le Théorème 0.2 et cette généralisation restent vrais [2] pour $n > p+1$, mais il faut alors remplacer les hyperplans de coordonnées par les « sous-espaces fortement invariants » [3] de l'action U : en nombre fini, ce sont les sous-espaces instables des différents champs de vecteurs $U(v)$. La démonstration, qui utilise les idées du Chapitre 3 de [3], est plus complexe que celle donnée ci-après pour $n = p+1$.

Les Éqs. (E_j) jouent un rôle essentiel dans la linéarisation des structures de Poisson [4,1,2].

Idée de la preuve du Théorème 0.2

Pour $\epsilon > 0$ assez petit, $Q_p := \{x \in \mathbf{R}^n\colon\ |x_2| = \cdots = |x_n| = \epsilon\}$ coupe chaque orbite de la restriction à $\mathbf{R}^n \setminus \{x_2 \cdots x_n = 0\}$ de l'action linéaire $(t, x) \mapsto \mathrm{e}^{U(t)} x$ de $\mathbf{R}^p$ sur $\mathbf{R}^n$ transversalement en un seul point. En utilisant p fois le Théorème 0.1, nous montrons que *tout* germe f_p en $\{x_1 = 0\}$ de fonction C^∞ sur Q_p s'annulant à tous les ordres en $\{x_1 = 0\}$ est la restriction d'une unique solution f de notre problème.

1. Introduction

Hypotheses. Let $h:(M,\Sigma)\to(M,\Sigma)$ be a germ at $\Sigma:=\mathbf{T}^r\times\{(0,0,0)\}$ of a C^∞-diffeomorphism of the manifold $M:=\mathbf{T}^r\times\mathbf{R}^\ell\times\mathbf{R}^m\times\mathbf{R}^q$ given by $h(\theta,x,y,z)=(e(\theta,x,y),f_+(\theta,x,y),f_-(\theta,x,y),g(\theta,x,y,z))$. We suppose that the germs at Σ of $W^+:=\mathbf{T}^r\times\mathbf{R}^\ell\times\{(0,0)\}$ and $W^-:=\mathbf{T}^r\times\{0\}\times\mathbf{R}^m\times\{0\}$ are h-invariant and that the germ of $W:=\mathbf{T}^r\times\mathbf{R}^\ell\times\mathbf{R}^m\times\{0\}$ has an infinite contact with its image by h along $W^+\cup W^-$. Moreover, we assume ('normal hyperbolicity') that the linear automorphisms $A_\theta\in\mathrm{GL}(\mathbf{R}^\ell)$ and $B_\theta\in\mathrm{GL}(\mathbf{R}^m)$ given by $A_\theta:=\frac{\partial f_+}{\partial x}(\theta,0,0)$ and $B_\theta:=\frac{\partial f_-}{\partial y}(\theta,0,0)$ satisfy

$$\sup_\theta|A_\theta|<1\quad\text{and}\quad\sup_\theta\left|B_\theta^{-1}\right|<1.\tag{HN}$$

Then, we have the following extension result [1]:

Theorem 1.1. *Under those hypotheses, any h-invariant germ at Σ of a C^∞-submanifold V of $M\setminus W^-$ having infinite contact with W along W^+ has a unique extension to an h-invariant germ at Σ of a C^∞-submanifold $\widetilde{V}$ of M, which has infinite contact with W along W^-.*

Indications on the proof [1]

In a neighbourhood of $\{y=0\}$ in $M\setminus W^-$, the submanifold V is the graph of a C^∞ map $\phi:\mathbf{T}^r\times\mathbf{R}^\ell\times\mathbf{R}^m\to\mathbf{R}^q$ such that $\frac{\partial^j\phi}{\partial y^j}(\theta,x,0)=0$ for all $j\in\mathbf{N}$. The proof consists in establishing that, in a neighbourhood of $\{(x,y)=(0,0)\}$ *in M*, the submanifold V can be extended in a unique way to the h-invariant graph of a C^∞ map (again denoted by ϕ), and that $\frac{\partial^j\phi}{\partial x^j}(\theta,0,y)=0$ for all $j\in\mathbf{N}$.

Remark 1. The closure of the definition domain of the initial ϕ contains a priori no open subset of W^-. Thus, uniqueness comes from the fact we wish to construct an h-invariant graph.

The proof of Theorem 1.1, which will be included in a more developed paper [2], looks like the proof of the theorem in Section (A.6.1) of [3], Appendix 6. The latter plays a central role in the linearisation of germs of $\mathbf{R}^p$ actions ([3], Chapter 3); it follows at once from Theorem 0.1, and so do the statements in sections (A.6.3) and (A.6.5) of [3], Appendix 6.

Instead of these applications, we shall prove a result which cannot be found in [3].

1.1. An application of the extension theorem

Given an integer $n\geqslant 2$, set $p:=n-1$ and consider p linear diagonal vector fields $U_1,\dots,U_p$ on $\mathbf{R}^n$; in particular, $[U_i,U_j]=0$ for $1\leqslant i<j\leqslant p$. If we denote by $U:\mathbf{R}^p\to\mathrm{gl}_n(\mathbf{R})$ the linear map $v\mapsto\sum_{i=1}^p v_iU_i$, we have that $U(v)=\sum_{i=1}^n\lambda_i(v)x_i\frac{\partial}{\partial x_i}$. Suppose that the linear forms $\lambda_i:\mathbf{R}^p\to\mathbf{R}$ satisfy the following hypotheses:

(H1) The infinitesimal action U is *hyperbolic*, i.e. any p of the λ_i's are linearly independent.
(H2) It is in the 'Siegel domain', i.e. the convex enveloppe of the λ_i's contains the origin.

Given germs $\theta_j:(\mathbf{R}^n,0)\to\mathbf{R}$ of C^∞ functions, we wish to solve the equations

$$\mathcal{L}_{U_j}f=\theta_j\quad\text{for }1\leqslant j\leqslant p.\tag{E$_j$}$$

Theorem 1.2. *If the θ_j's satisfy the obviously necessary conditions $\mathcal{L}_{U_j}\theta_i=\mathcal{L}_{U_i}\theta_j$ for $1\leqslant i<j\leqslant p$ and vanish at every order along the coordinate hyperplanes, then the germs $f:(\mathbf{R}^n,0)\to\mathbf{R}$ which vanish at all orders along the coordinate hyperplanes and solve the equations (E_j) for $1\leqslant j\leqslant p$ form an infinite dimensional affine space.*

Remark 2. This implies the analogous theorem for generic germs U_j of mutually commuting vector fields vanishing at $0\in\mathbf{R}^n$, for they can be simultaneously C^∞-linearised [3].

Theorem 1.2 (and this generalisation) remain true for $n > p+1$, provided the coordinate hyperplanes are replaced by the 'strongly invariant subspaces' [3] of the action U: these are the unstable subspaces of the various vector fields $U(v)$, and they form a finite set. The proof [2], which uses the ideas of Chapter 3 in [3], is more complex than for $n = p+1$.

The equations (E_j) play a key role in the linearisation of Poisson structures [4,1,2].

2. Proof of Theorem 1.2

2.1. Choice of a good presentation of the action; 'quasi-quotients'

Setting $\Theta(v) := v_1\theta_1 + \cdots + v_p\theta_p$, the problem we wish to solve is equivalent to

$$\forall v \in \mathbf{R}^p \quad \mathcal{L}_{U(v)} f = \Theta(v)$$

and therefore, in (E_j), we can take $U_1 := U(e_1), \ldots, U_p := U(e_p)$ for any basis $(e_1, \ldots, e_p)$ of $\mathbf{R}^p$.

As the action is hyperbolic, the hypothesis $0 \in \mathrm{Conv}(\lambda_1, \ldots, \lambda_n)$ reads

$$0 = \sum_{i=1}^{n} \alpha_i \lambda_i \quad \text{with } \alpha_1 > 0, \ldots, \alpha_n > 0. \tag{1}$$

We choose the basis $(e_1, \ldots, e_p)$ of $\mathbf{R}^p$ as follows: e_k is determined by the conditions

$$\lambda_1(e_k) = \cdots = \lambda_k(e_k) = 1 \quad \text{and} \quad \lambda_j(e_k) = 0 \quad \text{for } k+1 < j \leqslant n,$$

which imply $\lambda_{k+1}(e_k) < 0$ because of (1). To check that the vectors $e_1, \ldots, e_p$ are linearly independent, assume $\sum_{k=1}^{p} \mu_k e_k = 0$ and apply $\lambda_n, \ldots, \lambda_2$ to this equality to get $\mu_p = \cdots = \mu_1 = 0$.

Given some small $\epsilon > 0$, and setting $Q_d := \{x \in \mathbf{R}^n\colon |x_j| = \epsilon \text{ for } n-d < j \leqslant n\}$, $0 \leqslant d \leqslant p$ (hence $Q_0 = \mathbf{R}^n$), we have the following: for $1 \leqslant k \leqslant p$, the vector field $U_k := U(e_k)$ is tangent to Q_{p-k} and each orbit of its restriction to $K_{p-k+1} := Q_{p-k} \smallsetminus \{x_{n-k} = 0\}$ intersects Q_{p-k+1} transversally, at exactly one point (thus, Q_{p-k+1} is a realisation of the quotient of K_{p-k+1} by the action of $\mathbf{R}$ generated by U_k).

As the U_j's commute, they generate the linear action $(t, x) \mapsto \mathrm{e}^{U(t)}x$ of $\mathbf{R}^p$ on $\mathbf{R}^n$ and each Q_d with $d > 0$ is the quotient of $\mathbf{R}^n \smallsetminus \{x_{n-d+1} \cdots x_n = 0\}$ by the linear action of $\mathbf{R}^d$ generated by our new $U_1, \ldots, U_d$.

We shall prove inductively that, if ϵ is small enough, *any* germ f_p at $\{x_1 = 0\}$ of a smooth function on Q_p vanishing at all orders at $\{x_1 = 0\}$ is the restriction of a unique solution f of our problem.

2.2. End of the proof

Given an integer $k < p$, we make the induction hypothesis that f_p has a unique extension to a germ f_{p-k} at $\{x_1 = \cdots = x_{k+1} = 0\}$ of a smooth function on Q_{p-k}, satisfying

$$\mathcal{L}_{U(e_j)} f = \Theta(e_j) \quad \text{for } 1 \leqslant j \leqslant k$$

and vanishing at all orders along the intersections of the coordinate hyperplanes with Q_{p-k}. By the very definition of f_p, this is satisfied if $k = 0$. We shall show that, if ϵ is small enough, the Cauchy problem

$$\begin{cases} \mathcal{L}_{U_{k+1}} f = \Theta_{k+1} := \Theta(e_{k+1}), \\ f|_{Q_{p-k}} = f_{p-k}, \end{cases} \tag{2}$$

(where $U_{k+1} = U(e_{k+1})$) has a unique solution in Q_{p-k-1}, whose germ f_{p-k-1} at $\{x_1 = \cdots = x_{k+2} = 0\}$ satisfies the induction hypothesis with $k := k+1$, yielding Theorem 1.2 after p steps.

As Q_{p-k-1} is the disjoint union of the affine $(k+2)$-planes in $\mathbf{R}^n$ parallel to $\mathbf{R}^{k+2} \times \{0\}$ and passing through the points of $\Sigma_{p-k-1} := \{x \in \mathbf{R}^n\colon x_j = 0 \text{ for } j \leqslant k+2 \text{ and } x_j = \pm\epsilon \text{ otherwise}\}$, we may assume ϵ small enough for Θ_{k+1} to be defined in a neighbourhood O_{p-k-1} of each point o_{p-1} of Σ_{p-k-1} in Q_{p-k-1}, and we shall apply Theorem 1.1 to the germ $h^t_{k+1} : (Q_{p-k-1} \times \mathbf{R}, (o_{p-k-1}, 0)) \to (Q_{p-k-1} \times \mathbf{R}, (o_{p-1}, 0))$ given by

$$h^t_{k+1}(x, y) = \left(g^t_{k+1}(x), y + \int_0^t \Theta_{k+1}\bigl(g^s_{k+1}(x)\bigr)\,\mathrm{d}s \right)$$

for some fixed $t > 0$, where g_{k+1}^t denotes the flow of $U_{k+1}|_{Q_{p-k-1}}$. In the coordinate system $(x_1, \dots, x_{k+2})$, this flow reads $g_{k+1}^t(x) = (\mathrm{e}^{t\lambda_1(e_{k+1})}x_1, \dots, \mathrm{e}^{t\lambda_{k+2}(e_{k+1})}x_{k+2}) = (\mathrm{e}^t x_1, \dots, \mathrm{e}^t x_{k+1}, \mathrm{e}^{t\lambda_{k+2}(e_{k+1})}x_{k+2})$. Thus, since we have $\lambda_{k+2}(e_{k+1}) < 0$, the hypotheses of the Theorem 1.1 are satisfied by h_{k+1}^t with $r = 0$, $\ell = k+1$, $\ell = q = 1$, $A_\theta = \mathrm{e}^{t\lambda_{k+2}(e_{k+1})}$ and $B_\theta = \mathrm{e}^t$ if we put the $(k+2)$-nd factor in first position. Thus, W_+ is the x_{k+2}-axis and $W^- = \{x_{k+2} = y = 0\}$. The germ V to be extended is the (germ at 0 of the) graph of the smooth function f defined in a neighbourhood of $W_+ \setminus \{0\}$ in $Q_{p-k-1} \setminus W^-$ as the solution of the Cauchy problem (2). Indeed, if $\tau : Q_{p-k-1} \setminus W^- \to \mathbf{R}$ is given by $g_{k+1}^{-\tau(x)}(x) \in Q_{p-k}$ (there is a simple explicit formula!), this solution is $f(x) = f_{p-k}(g_{k+1}^{-\tau(x)}(x)) + \int_0^{\tau(x)} \Theta_{k+1}(g_{k+1}^{s-\tau(x)}(x))\,\mathrm{d}s$. Applying the extension theorem, we get a submanifold $\widetilde{V}$ which is the graph of our function f_{p-k-1}.

This function vanishes at all orders along the coordinate hyperplanes partly because of the extension theorem and partly because f_{p-k} and Θ_{k+1} do. It satisfies $\mathcal{L}_{U_j} f = \Theta_j := \Theta(e_j)$ for $1 \leqslant j \leqslant k$ since f_{p-k} does, $[U_{k+1}, U_j] = 0$ and $\mathcal{L}_{U_j}\Theta_{k+1} = \mathcal{L}_{U_{k+1}}\Theta_j$.

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