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Numerical Analysis

Relationship between multiscale enrichment and stabilized finite element methods for the generalized Stokes problem

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Abstract

We derive a new stabilized finite element method for the generalized Stokes problem starting from the non-stable continuous $\mathbb{P}_1/\mathbb{P}_1$ finite element space enriched with multiscale functions. The stabilization parameter is related with the enrichment functions which are analytically computed from a boundary value problem at the element level leading to a method which is free of constants. Optimal error estimates are obtained in natural norms and numerical tests validate the method. ***To cite this article: G.R. Barrenechea, F. Valentin, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Relation entre l'enrichisement multi-échelles et les méthodes d'éléments finis stabilisées pour le problème de Stokes généralisé. On propose une nouvelle méthode d'éléments finis stabilisée pour le problème de Stokes généralisé basée sur l'enrichissement de l'espace d'éléments finis continu $\mathbb{P}_1/\mathbb{P}_1$ par des fonctions multi-échelles. Le paramètre de stabilisation est donné par la moyenne de la fonction d'enrichissement sur l'élément, qui à son tour est calculée analytiquement par la résolution d'un problème aux limites dans chaque élément. Des estimations d'erreurs optimales sont obtenues et des tests numériques sont présentés. ***Pour citer cet article : G.R. Barrenechea, F. Valentin, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Dans ce travail, on s'intéresse au développement et à l'analyse numérique d'une nouvelle méthode stabilisée (9) pour le problème de Stokes généralisé. On étend la technique d'enrichissement des espaces polynomiaux proposée dans [4], de façon à enrichir l'espace d'éléments finis continu non stable $\mathbb{P}_1/\mathbb{P}_1$ par de nouvelles fonctions multi-échelles, solutions de (12), (13). Ensuite, on montre l'équivalence entre cette approche et une méthode d'éléments

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finis stabilisée, où le paramètre de stabilisation est obtenu analytiquement par la moyenne de la fonction multi-échelle sur l'élément (18). Pour cela, on utilise des projections L^2 locales sur l'espace des fonctions constantes. On démontre que le problème discret est bien posé dans le Lemme 4.1, et des estimations optimales dans les normes naturelles sont presentées dans le Théorème 4.2. Finalement, on valide les résultats théoriques par un test numérique qui démontre la précision et la stabilité de la méthode (Fig. 1).

1. Introduction

Numerical methods for the generalized Stokes problem based on the finite element method are limited by the compatibility condition (or inf-sup condition) between velocity and pressure spaces [3]. On the other hand, spurious oscillations may also appear due to the singular perturbation in the reaction-dominated regime. It is well known that stabilized finite element methods applied to the Stokes operator allow us to adopt equal order pair of spaces for velocity and pressure even if they do not satisfy the inf-sup condition. Some light on the origin of such methods, as well as on the design of stabilization parameters, has been proposed in the last ten years, and was mainly based on equivalence to enriching classical spaces with bubble functions. The theoretical framework of such analogy was derived in [1] for the Stokes equations, and furthermore extended for the generalized Stokes model in [2]. However, the stabilized finite element methods arising from bubble condensation presented an important drawback, namely the fact that the bubble function to be condensed was not known analytically, and hence the condensation procedure led to a stabilization parameter that was not known exactly. In order to correct this drawback, in this work we extend the Petrov–Galerkin approach introduced in [4] to the generalized Stokes problem. Beginning by enriching the $\mathbb{P}_1/\mathbb{P}_1$ continuous space with multiscale functions which are no longer bubble-like ones, and performing static condensation, we develop a new stabilized finite element method containing a stabilization parameter which is exactly known.

The outline of this Note is as follows. Section 2 includes the model and the enrichment strategy. The stabilized method is proposed and derived in Section 3 and analyzed in Section 4. Finally, in Section 5 we present a numerical validation of the proposed method.

2. The model problem and multiscale enrichment

Let Ω be an open bounded domain in $\mathbb{R}^2$ with polygonal boundary, $\boldsymbol{f} \in L^2(\Omega)^2$ and let us consider the following generalized Stokes problem: Find $(\boldsymbol{u}, p)$ such that

$$\begin{aligned} &\mathcal{L}\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \nabla\cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \\ &\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\mathcal{L}\boldsymbol{u} := \sigma\boldsymbol{u} - \nu\Delta\boldsymbol{u}$, and $\sigma, \nu \in \mathbb{R}^+$ denote the reaction term and the fluid viscosity, respectively. Let now $\{\mathcal{T}_h\}_{h>0}$ be a family of regular triangulations of Ω, build up using triangles K with boundary $\partial K = F_1 \cup F_2 \cup F_3$, $h_K :=$ diam(K) and $h := \max\{h_K\colon K \in \mathcal{T}_h\}$. Let

$$V_h = \bigl\{v \in C^0(\overline{\Omega})\colon v|_K \in \mathbb{P}_1(K),\ \forall K \in \mathcal{T}_h\bigr\},$$

and $\mathbf{V}_h := [V_h \cap H_0^1(\Omega)]^2$, $Q_h := V_h \cap L_0^2(\Omega)$ be the finite element spaces used to approximate velocity and pressure, respectively. Let $H^1(\mathcal{T}_h)$ and $H_0^1(\mathcal{T}_h)$ be the spaces of functions whose restriction to $K \in \mathcal{T}_h$ belongs to $H^1(K)$ and $H_0^1(K)$, respectively. Furthermore, $(\cdot,\cdot)_D$ stands for the inner product in $L^2(D)$ (or in $L^2(D)^2$, when necessary), and we denote by $\|\cdot\|_{s,D}$ ($|\cdot|_{s,D}$) the norm (seminorm) in $H^s(D)$ (or $H^s(D)^2$, if necessary).

In order to propose the Petrov–Galerkin method for Stokes problem (1), let $E_h \subset H^1(\mathcal{T}_h)$ be a finite dimensional space, called multiscale space, such that $V_h \cap E_h = \{0\}$, and we will only suppose by now that E_h is such that problem (3) below admits a solution. Then, our scheme reads: *Find* $\boldsymbol{u}_1 \in \mathbf{V}_h$, $\boldsymbol{u}_e \in [E_h]^2$ *and* $p_1 \in Q_h$ *such that*

$$a(\boldsymbol{u}_1 + \boldsymbol{u}_e, \boldsymbol{v}_h) - (p_1, \nabla\cdot \boldsymbol{v}_h)_\Omega + \bigl(q_1, \nabla\cdot(\boldsymbol{u}_1 + \boldsymbol{u}_e)\bigr)_\Omega = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega,$$

for all $\boldsymbol{v}_h \in \mathbf{V}_h \oplus [H_0^1(\mathcal{T}_h)]^2$ and all $q_1 \in Q_h$, where $a(\boldsymbol{u}, \boldsymbol{v}) := \sigma(\boldsymbol{u}, \boldsymbol{v})_\Omega + \nu(\nabla\boldsymbol{u}, \nabla\boldsymbol{v})_\Omega$. Now, this Petrov–Galerkin scheme is equivalent to the following system:

$$a(\boldsymbol{u}_1 + \boldsymbol{u}_e, \boldsymbol{v}_1) - (p_1, \nabla\cdot \boldsymbol{v}_1)_\Omega + \bigl(q_1, \nabla\cdot(\boldsymbol{u}_1 + \boldsymbol{u}_e)\bigr)_\Omega = (\boldsymbol{f}, \boldsymbol{v}_1)_\Omega \quad \forall(\boldsymbol{v}_1, q_1) \in \mathbf{V}_h \times Q_h, \tag{2}$$

$$a(\boldsymbol{u}_1 + \boldsymbol{u}_e, \boldsymbol{v}_b)_K - (p_1, \nabla\cdot \boldsymbol{v}_b)_K = (\boldsymbol{f}, \boldsymbol{v}_b)_K \quad \forall \boldsymbol{v}_b \in H_0^1(K)^2,\ \forall K \in \mathcal{T}_h, \tag{3}$$

where the subindex K stands for integration over K. Eq. (3) above may be written in strong form in the following way

$$\mathcal{L}\boldsymbol{u}_e = \boldsymbol{f} - (\mathcal{L}\boldsymbol{u}_1 + \nabla p_1) \quad \text{in } K. \tag{4}$$

Now, this differential problem above must be completed with boundary conditions. In order to correct also the residual of the strong equation in the boundary of K, we impose the following boundary condition on $\boldsymbol{u}_e$:

$$\boldsymbol{u}_e = \boldsymbol{g}_e \quad \text{on } F_i,\ i = 1, 2, 3, \tag{5}$$

where $\boldsymbol{g}_e$ is the solution of

$$\bar{\sigma}_i \boldsymbol{g}_e - \nu\partial_{ss}\boldsymbol{g}_e = \boldsymbol{f} - (\mathcal{L}\boldsymbol{u}_1 + \nabla p_1) \quad \text{in } F_i, \qquad \boldsymbol{g}_e = \boldsymbol{0} \quad \text{on the nodes}, \tag{6}$$

where $\bar{\sigma}_i$, $i = 1, 2, 3$, is independent of h, and will be specified later, and ∂_s denotes the tangential derivative operator. In this way, we can define an operator $\mathcal{M}_K : L^2(K)^2 \to H^1(K)^2$ such that

$$\boldsymbol{u}_e^K := \boldsymbol{u}_e|_K = \mathcal{M}_K(\boldsymbol{f} - \mathcal{L}\boldsymbol{u}_1 - \nabla p_1) \quad \forall K \in \mathcal{T}_h, \tag{7}$$

thus with the characterization (7), problem (2) reads

$$\sum_{K\in\mathcal{T}_h}\Big[a\big(\boldsymbol{u}_1 - \mathcal{M}_K(\mathcal{L}\boldsymbol{u}_1 + \nabla p_1), \boldsymbol{v}_1\big)_K - (p_1, \nabla\cdot\boldsymbol{v}_1)_K + \big(q_1, \nabla\cdot\big(\boldsymbol{u}_1 - \mathcal{M}_K(\mathcal{L}\boldsymbol{u}_1 + \nabla p_1)\big)\big)_K\Big]$$
$$= (\boldsymbol{f}, \boldsymbol{v}_1)_\Omega - \sum_{K\in\mathcal{T}_h}\Big[a(\mathcal{M}_K\boldsymbol{f}, \boldsymbol{v}_1)_K - \big(q_1, \nabla\cdot(\mathcal{M}_K\boldsymbol{f})\big)_K\Big] \quad \forall(\boldsymbol{v}_1, q_1) \in \mathbf{V}_h \times Q_h. \tag{8}$$

We first remark that (7) provides a precise definition for space E_h, indeed, we can now write $[E_h]^2 := \{\boldsymbol{v} \in H^1(\mathcal{T}_h)^2\colon\ \exists\boldsymbol{v}_1 \in \mathbb{P}_1(K)^2,\ \boldsymbol{v} = \mathcal{M}_K(\boldsymbol{v}_1)\}$, which clearly is a finite dimensional space. Next, we remark that the method (8) is clearly consistent. From now on, we will suppose that this method leads to a well posed problem. For the reaction–diffusion equation this fact has been rigorously justified in [4] and we will address this issue in a future work. Here, we are interested in the formal derivation of a stabilized finite element method from (8) and in its numerical analysis.

3. The stabilized finite element method

We begin by presenting the stabilized finite element method: Find $(\boldsymbol{u}_1, p_1) \in \mathbf{V}_h \times Q_h$ such that

$$\mathbf{B}\big((\boldsymbol{u}_1, p_1), (\boldsymbol{v}_1, q_1)\big) = \mathbf{F}(\boldsymbol{v}_1, q_1) \quad \forall(\boldsymbol{v}_1, q_1) \in \mathbf{V}_h \times Q_h, \tag{9}$$

where

$$\mathbf{B}\big((\boldsymbol{u}_1, p_1), (\boldsymbol{v}_1, q_1)\big) := a(\boldsymbol{u}_1, \boldsymbol{v}_1) - (p_1, \nabla\cdot\boldsymbol{v}_1)_\Omega + (q_1, \nabla\cdot\boldsymbol{u}_1)_\Omega - \sum_{K\in\mathcal{T}_h}\tau_K(\mathcal{L}\boldsymbol{u}_1 + \nabla p_1, \mathcal{L}\boldsymbol{v}_1 - \nabla q_1)_K,$$

$$\mathbf{F}(\boldsymbol{v}_1, q_1) := (\boldsymbol{f}, \boldsymbol{v}_1)_\Omega - \sum_{K\in\mathcal{T}_h}\tau_K(\boldsymbol{f}, \mathcal{L}\boldsymbol{v}_1 - \nabla q_1)_K,$$

and the stabilization parameter is given by (denoting by F_i the edge opposed to the node i)

$$\tau_K := \frac{1}{\sigma}\left[1 - \sum_{i=1}^{3}\left(\frac{1}{\alpha_i^2} - \frac{1}{\alpha_i\sinh(\alpha_i)}\right)\right], \quad \text{where } \alpha_i := \sqrt{\frac{4\sigma|K|^2}{\nu|F_i|^2}}. \tag{10}$$

3.1. Derivation of the method

In this section, we intend to perform the formal derivation of the stabilized method (9). First, for simplicity we will suppose that $\boldsymbol{f}$ is a piecewise constant function and that $\mathcal{T}_h$ contains only equilateral triangles, even if the method (9) was proposed and will be analyzed for general $\boldsymbol{f}$ and regular $\mathcal{T}_h$. The first step is to replace in our formulation $\boldsymbol{u}_e$ by

$$\tilde{\boldsymbol{u}}_e := \mathcal{M}_K(\boldsymbol{f} - \mathcal{L}\bar{\boldsymbol{u}}_1 - \nabla p_1) = b_K(\boldsymbol{f} - \mathcal{L}\bar{\boldsymbol{u}}_1 - \nabla p_1), \tag{11}$$

where, for a function v, $\bar{v}$ denotes its projection onto the $\mathbb{P}_0(K)$ space, i.e., $\bar{v} := (v, 1)_K/|K|$, and from (7), b_K is the solution of the reaction–diffusion problem

$$\mathcal{L} b_K = 1 \quad \text{in } K, \qquad b_K = g \quad \text{on } \partial K, \tag{12}$$

where, for $i = 1, 2, 3$,

$$\bar{\sigma}_i g - \nu \partial_{ss} g = 1 \quad \text{in } F_i, \qquad g = 0 \quad \text{on the nodes.} \tag{13}$$

We further remark that, since $\bar{\sigma}_i$ does not depend on h, this function b_K satisfies

$$\|b_K\|_{0,K} \leqslant C h_K^3 \quad \text{and} \quad \|b_K\|_{0,\partial K} \leqslant C h_K^{5/2}, \tag{14}$$

where $C > 0$ is a positive constant depending possibly on σ and ν, but independent of h.

Next, in order to design a stabilized finite element method we integrate by parts and we have on each $K \in \mathcal{T}_h$,

$$\nu(\nabla \tilde{\boldsymbol{u}}_e, \nabla \boldsymbol{v}_1)_K = -\nu(\tilde{\boldsymbol{u}}_e, \Delta \boldsymbol{v}_1)_K + (\tilde{\boldsymbol{u}}_e, \nu \partial_{\boldsymbol{n}} \boldsymbol{v}_1)_{\partial K}, \qquad (q_1, \nabla\cdot \tilde{\boldsymbol{u}}_e)_K = -(\tilde{\boldsymbol{u}}_e, \nabla q_1)_K + (\tilde{\boldsymbol{u}}_e, q_1 \mathbf{I}\cdot \boldsymbol{n})_{\partial K},$$

where $\mathbf{I}$ is the $\mathbb{R}^{2\times 2}$ identity matrix, and using these identities we can rewrite (2) in the following way

$$a(\boldsymbol{u}_1, \boldsymbol{v}_1) - (p_1, \nabla\cdot \boldsymbol{v}_1)_\Omega + (q_1, \nabla\cdot \boldsymbol{u}_1)_\Omega + \sum_{K \in \mathcal{T}_h} \big[(\tilde{\boldsymbol{u}}_e, \mathcal{L} \boldsymbol{v}_1 - \nabla q_1)_K + (\tilde{\boldsymbol{u}}_e, \nu \partial_{\boldsymbol{n}} \boldsymbol{v}_1 + q_1 \mathbf{I}\cdot \boldsymbol{n})_{\partial K}\big]$$
$$= (\boldsymbol{f}, \boldsymbol{v}_1)_\Omega. \tag{15}$$

Now, we will see that we can actually neglect the boundary terms since they are of the order of the method. Indeed, we first remark that from the definition of $\bar{\sigma}_i$ (see below), then if the triangles are equilateral then $(b_K, 1)_{F_i}/|F_i| = (b_K, 1)_{\partial K}/|\partial K|$. Using this fact, and $(1, \partial_{\boldsymbol{n}} v)_{\partial K} = 0$ (for $v \in \mathbb{P}_1(K)$), it turns out that

$$(\tilde{\boldsymbol{u}}_e, \nu \partial_{\boldsymbol{n}} \boldsymbol{v}_1)_{\partial K} = \sum_{i=1}^{3} \frac{(b_K, 1)_{F_i}}{|F_i|} (\boldsymbol{f} - \mathcal{L} \bar{\boldsymbol{u}}_1 - \nabla p_1, \nu \partial_{\boldsymbol{n}} \boldsymbol{v}_1)_{F_i} = \frac{(b_K, 1)_{\partial K}}{|\partial K|} (\boldsymbol{f} - \mathcal{L} \bar{\boldsymbol{u}}_1 - \nabla p_1, \nu \partial_{\boldsymbol{n}} \boldsymbol{v}_1)_{\partial K} = 0,$$

which shows that the first boundary term in (15) vanishes. In order to bound the other term, let $\bar{q}_1$ be the projection of q_1 in K, then, using the approximation properties of the projection (see [3]), since $(\tilde{\boldsymbol{u}}_e, \bar{q}_1 \mathbf{I}\cdot \boldsymbol{n})_{\partial K} = 0$, and from (14) and Cauchy–Schwarz's inequality we arrive at

$$(\tilde{\boldsymbol{u}}_e, q_1 \mathbf{I}\cdot \boldsymbol{n})_{\partial K} = \big(b_K(\boldsymbol{f} - \mathcal{L} \bar{\boldsymbol{u}}_1 - \nabla p_1), (q_1 - \bar{q}_1) \mathbf{I}\cdot \boldsymbol{n}\big)_{\partial K} \leqslant C h_K^2 \|\boldsymbol{f} - \mathcal{L} \bar{\boldsymbol{u}}_1 - \nabla p_1\|_{0,K} |q_1|_{1,K}.$$

Using (14) again and analogous arguments, we have that $(\tilde{\boldsymbol{u}}_e, \sigma(\boldsymbol{v}_1 - \bar{\boldsymbol{v}}_1))_K \leqslant C h_K^3 \|\boldsymbol{f} - \mathcal{L} \bar{\boldsymbol{u}}_1 - \nabla p_1\|_{0,K} |\boldsymbol{v}_1|_{1,K}$, and hence, using (11) and the orthogonality of the projection the following approximation is justified

$$\sum_{K \in \mathcal{T}_h} (\tilde{\boldsymbol{u}}_e, \mathcal{L} \boldsymbol{v}_1 - \nabla q_1)_K \approx \sum_{K \in \mathcal{T}_h} (\tilde{\boldsymbol{u}}_e, \mathcal{L} \bar{\boldsymbol{v}}_1 - \nabla q_1)_K = \sum_{K \in \mathcal{T}_h} \frac{(b_K, 1)_K}{|K|} (\boldsymbol{f} - \mathcal{L} \bar{\boldsymbol{u}}_1 - \nabla p_1, \mathcal{L} \boldsymbol{v}_1 - \nabla q_1)_K.$$

Collecting all the previous results, we can present the following stabilized finite element method for (1): Find $(\boldsymbol{u}_1, p_1) \in \mathbf{V}_h \times Q_h$ such that

$$a(\boldsymbol{u}_1, \boldsymbol{v}_1) - (p_1, \nabla\cdot \boldsymbol{v}_1)_\Omega + (q_1, \nabla\cdot \boldsymbol{u}_1)_\Omega - \sum_{K \in \mathcal{T}_h} \frac{(b_K, 1)_K}{|K|} (\mathcal{L} \bar{\boldsymbol{u}}_1 + \nabla p_1, \mathcal{L} \boldsymbol{v}_1 - \nabla q_1)_K$$
$$= (\boldsymbol{f}, \boldsymbol{v}_1)_\Omega - \sum_{K \in \mathcal{T}_h} \frac{(b_K, 1)_K}{|K|} (\boldsymbol{f}, \mathcal{L} \boldsymbol{v}_1 - \nabla q_1)_K \quad \forall (\boldsymbol{v}_1, q_1) \in \mathbf{V}_h \times Q_h. \tag{16}$$

Since we want to present a stabilized finite element method with a classical structure, we replace the added terms in K by $(\mathcal{L} \boldsymbol{u}_1 + \nabla q_1, \mathcal{L} \boldsymbol{v}_1 - \nabla q_1)_K$, which introduces a new source of error, again of a smaller size.

We observe that in order to fully present the method it only remains to give an expression for the stabilization parameter τ_K. Let ψ_i, $i = 1, 2, 3$, be the barycentric coordinates of the element K. We first remark that the edges F_i, $i = 1, 2, 3$, are numbered such that $\psi_i|_{F_i} = 0$. Then, clearly $b_K = \sum_{i=1}^3 b_K^i$ where b_K^i is the solution of

$$\mathcal{L} b_K^i = \psi_i \quad \text{in } K,$$
$$\bar{\sigma}_i b_K^i - \nu \partial_{ss} b_K^i = \psi_i \quad \text{in } F_i, \qquad b_K^i = 0 \quad \text{on the nodes,}$$

where, suggested by [4], we have made the choice $\bar{\sigma}_i = \sigma(\partial_s \psi_i)^2/\gamma_i$, where $\gamma_i = (\partial\psi_i/\partial x)^2 + (\partial\psi_i/\partial y)^2 = |F_i|^2/(4|K|^2)$. Let us choose $\lambda_i := \psi_i - \sigma b_K^i$. Then, λ_i, $i = 1, 2, 3$, solves

$$\mathcal{L}\lambda_i = 0 \quad \text{in } K,$$
$$\bar{\sigma}_i \lambda_i - \nu \partial_{ss}\lambda_i = 0 \quad \text{in } F_i, \qquad \lambda_i = \psi_i \quad \text{on the nodes,}$$

and it turns out that this is the same problem from [4] with solution

$$\lambda_i(x, y) = \frac{\sinh(\alpha_i \psi_i)}{\sinh(\alpha_i)} \quad \text{where } \alpha_i = \sqrt{\frac{\sigma}{\nu\gamma_i}}. \tag{17}$$

Finally, from the last computations we see that the stabilization parameter τ_K is given by

$$\tau_K = \frac{(b_K, 1)_K}{|K|} = \frac{1}{\sigma|K|}\left(1 - \sum_{i=1}^{3}\lambda_i, 1\right)_K = \frac{1}{\sigma}\left[1 - \frac{1}{|K|}\sum_{i=1}^{3}(\lambda_i, 1)_K\right], \tag{18}$$

and since we have an explicit solution for λ_i, last integrals may be calculated analytically, obtaining the expression (10) for τ_K.

4. Stability and convergence analysis

Before performing the stability analysis, let us remark that, using (14), the stabilization parameter τ_K satisfies $\tau_K \leqslant C h_K^2$, where the constant C may depend on σ and ν, but not on h. Next, we define the following mesh dependent norm

$$\left\|(\boldsymbol{v}_1, q_1)\right\|_h^2 := \sum_{K \in \mathcal{T}_h}\left[\sigma(1 - \sigma\tau_K)\|\boldsymbol{v}_1\|_{0,K}^2 + \nu|\boldsymbol{v}_1|_{1,K}^2 + \tau_K|q_1|_{0,K}^2\right], \tag{19}$$

and using the fact that $1 - \sigma\tau_K > 0$ in each $K \in \mathcal{T}_h$, we can state the following stability result.

Lemma 4.1. *The discrete problem* (9) *has a unique solution since the bilinear form* $\mathbf{B}$ *satisfies*

$$\mathbf{B}\big((\boldsymbol{v}_1, q_1), (\boldsymbol{v}_1, q_1)\big) = \left\|(\boldsymbol{v}_1, q_1)\right\|_h^2 \quad \forall(\boldsymbol{v}_1, q_1) \in \mathbf{V}_h \times Q_h.$$

Finally, using the previous result, appropriate interpolation inequalities (cf. [3]) and the asymptotic behavior of τ_K we can prove the following optimal convergence result.

Theorem 4.2. *Let us suppose that Ω is a convex polygon, that $(\boldsymbol{u}, p) \in [H^2(\Omega) \cap H_0^1(\Omega)]^2 \times (H^1(\Omega) \cap L_0^2(\Omega))$ is the solution of* (1) *and that $(\boldsymbol{u}_1, p_1) \in \mathbf{V}_h \times Q_h$ is the solution of* (9). *Then there exists $C > 0$, independent of h, such that*

$$\left\|(\boldsymbol{u} - \boldsymbol{u}_1, p - p_1)\right\|_h \leqslant Ch\big(|\boldsymbol{u}|_{2,\Omega} + |p|_{1,\Omega}\big), \qquad \|p - p_1\|_{0,\Omega} \leqslant Ch\big(|\boldsymbol{u}|_{2,\Omega} + |p|_{1,\Omega}\big),$$
$$\|\boldsymbol{u} - \boldsymbol{u}_1\|_{0,\Omega} \leqslant Ch^2\big(|\boldsymbol{u}|_{2,\Omega} + |p|_{1,\Omega}\big).$$

We finally remark that we have not explored at this stage the influence of physical constants on the error. This matter, as well as a detailed error analysis between the original Petrov–Galerkin method (8) and our actual stabilized finite element method (9), will be the subject of a future research.

5. Numerical experiments

We asses the lid-driven cavity problem, with domain $\Omega = (0, 1) \times (0, 1)$, $\boldsymbol{f} = \mathbf{0}$, and we perform experiments with $\nu = 1$ and $\nu = 10^{-5}$, both using $\sigma = 1$. We depict in Fig. 1 elevations for the pressure field (left, $\nu = 1$) and of the tangential velocity (right, $\nu = 10^{-5}$). We observe the absence of oscillations in both cases, which shows that the method prevents spurius oscillations of the pressure and captures correctly the boundary layers.

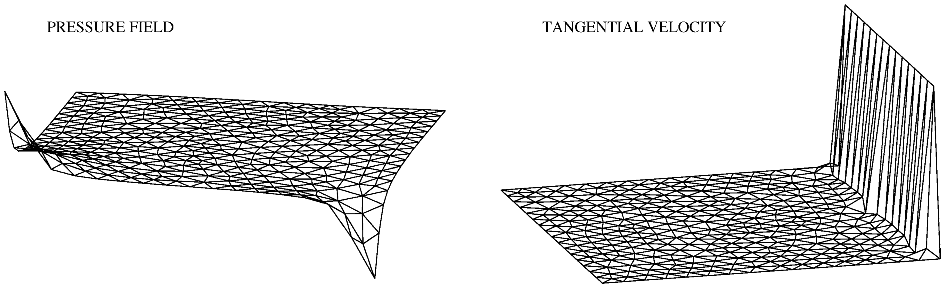

Fig. 1. Pressure field (left, $\nu = 1$) and tangential velocity (right, $\nu = 10^{-5}$).

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