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Stochastic covariant calculus of order two

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Abstract

We study continuous semimartingales in a vector fibre bundle E over a differentiable manifold M. Following Meyer and Schwartz's principle, we show that stochastic covariant integration on a manifold involves second order differential operators, according to the Itô formula integrating not only the differential but the 2-jet of a function, and using a notion of connection of order 2. A fundamental example of such a connection is given by the 2-jet of the parallel transport along geodesics on M. ***To cite this article: L. Maillard-Teyssier, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Calcul stochastique covariant d'ordre deux. Nous étudions des semimartingales continues à valeurs dans un fibré vectoriel E au dessus d'une variété différentielle M. Dans la continuité des travaux de Meyer et Schwartz, nous montrons que l'intégration stochastique sur une variété concerne en réalité des opérateurs différentiels d'ordre 2, en accord avec la formule d'Itô qui fait intervenir le 2-jet d'une fonction et non sa différentielle, et à l'aide de la notion de connexion d'ordre 2. Un exemple fondamental de connexion d'ordre 2 est donné par le 2-jet du transport parallèle le long des géodésiques sur M. ***Pour citer cet article : L. Maillard-Teyssier, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

1. Introduction and reminders

Let (z_t) be a continuous semimartingale on a manifold U. To integrate along (z_t), first order forms are usually considered. The transfer principle permits to define Stratonovich integrals on manifolds, then Itô calculus on U is usually deduced from a Stratonovich to Itô conversion formula, for which we need a torsion-free connection on U. However, the stochastic infinitesimal variations ∂z_t (Stratonovich) and dz_t (Itô) are not intrinsic. The Itô formula rather incites us to consider a second order stochastic infinitesimal variation, similar to a second order tangent

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vector on U, $\mathbf{d}z_t = \mathrm{d}z_t^i \frac{\partial}{\partial z^i} + \frac{1}{2}\mathrm{d}\langle z^i, z^j\rangle_t \frac{\partial^2}{\partial z^i \partial z^j}$. Therefore, integrands are second order forms on U (for instance the 2-jet $d^2 f$ of a C^2 function f on U). This set is denoted by $\tau^* U$. This second order formalism, providing compact and geometrically intrinsic formulas, was developed by Meyer in [6], [7] and [8] and by Schwartz in [11]. In [3], Emery writes Stratonovich and Itô integrals as particular cases of integrals of order 2. Maps need to be introduced, d_s for Stratonovich, G for Itô, allowing us to transform a first order form α into a second order form. With regard to the Itô integral, a torsion-free connection is needed on U, which induces the map G.

Definition 1.1. The Stratonovich integral of α along (z_t) is defined by $\int \alpha_z \partial z = \int (d_s\alpha)_z \,\mathbf{d}z$. In an imbedding, it is written as $\int (d_s\alpha)_{z_t} \,\mathbf{d}z_t = \int (\alpha_{z_t})_i \,\mathrm{d}z_t^i + \frac{1}{2}\int \frac{\partial \alpha_i}{\partial z^j}(z_t)\,\mathrm{d}\langle z^i, z^j\rangle_t$.

Definition 1.2. The Itô integral of α along (z_t) is defined by $\int \alpha_z \,\mathrm{d}z = \int (G\alpha)_z \,\mathbf{d}z$. In an imbedding, it is written, with the Christoffel symbols of G, as $\int (G\alpha)_{z_t} \,\mathbf{d}z_t = \int (\alpha_{z_t})_i \,\mathrm{d}z_t^i + \frac{1}{2}\int (\alpha_{z_t})_k \Gamma_{ij}^k(z_t)\,\mathrm{d}\langle z^i, z^j\rangle_t$.

In the following, we will be interested in semimartingales in a vector fibre bundle E over M. For all V in E, we write $V = (x, v)$, with x in M and v in the fibre E_x through V. Let us recall the definition of a connection on M, following [4], [12] and [13]. The tangent space $T_V E$ to E at V contains a canonical subset $T_v(E_x)$, called the vertical space at V. A connection on M is a way of choosing for each V a supplementary space, called horizontal, of this vertical space.

Definition 1.3. A connection on M is a smooth map $p : V \in E \to p_V$ such that for all $V = (x, v)$ in E (i) p_V projects $T_V E$ onto the vertical space $T_v(E_x)$, (ii) $\forall \lambda \in \mathbb{R}$, $p_{\tilde{\lambda} V} \circ d\tilde{\lambda}(V) = \lambda p_V$, with $\tilde{\lambda}(V) = (x, \lambda.v)$.

Let $(Y_t) = ((x_t, y_t))$ be a C^1 curve in E. Its covariant derivative can be seen as the projection of the usual derivative onto the vertical space: $\frac{\partial^{\mathcal{V}} Y_t}{\partial t} = p_{Y_t}(\dot{Y}_t)$. In an imbedding, we write $\frac{\partial^{\mathcal{V}} Y_t}{\partial t} = [\dot{y}_t^i + \Gamma_{jk}^i(x_t) y_t^k \dot{x}_t^j]\frac{\partial}{\partial y^i}$, with $\frac{\partial}{\partial y^i} \in T_{y_t} E_{x_t}$, and where Γ_{jk}^i are the Christoffel symbols of p.

(*Notice our* (*non-usual*) *notation*: *the exponent* $\mathcal{V}$ *stands for* "*vertical*", *since* $\frac{\partial^{\mathcal{V}} Y_t}{\partial t}$ *is in the vertical space*.)

Norris [9] and Elworthy [2] have defined stochastic covariant continuous integrals. They are equivalently written as follows, using the form $p^*(\alpha)$ on E ($\forall W \in T_V E$, $p^*(\alpha)_V(W) = \alpha_V(p_V(W))$).

Definition 1.4. Let $(Y_t) = ((x_t, y_t))$ be a continuous semimartingale in E and α a form on M. The Stratonovich covariant integral of α along (Y_t) is defined by $\int \alpha_Y \partial^{\mathcal{V}} Y = \int p^*(\alpha)_Y \partial Y$.

The Itô covariant integral of α along (Y_t) is defined by $\int \alpha_Y \,\mathrm{d}^{\mathcal{V}} Y = \int p^*(\alpha)_Y \,\mathrm{d}Y$, where the Itô integration along Y is defined with a connection $\mathbb{G}$ on E (for instance the horizontal lift of the connection p).

We want to write these integrals as second order integrals, in order to understand stochastic covariant calculus as a second order calculus, according to Meyer and Schwartz's principle.

2. Stochastic covariant calculus of order two

We need to specify the notion of connection of order two on M. The tangent space of order two, $\tau_V E$, to E at V, contains a canonical subset $\tau_v(E_x)$. We call it the vertical space of order two at V. Thus, a connection of order two $\tilde{p}$ on M is a way of choosing for each V a supplementary horizontal space of this vertical space. Then, by analogy with the first order calculus, we can define the covariant calculus of order two as the image by $\tilde{p}$ of the (non-covariant) calculus of order two.

Definition 2.1. A connection of order two on M is a smooth map $\tilde{p}: V \in E \to \tilde{p}_V$ such that for all $V = (x, v)$ in E, (i) $\tilde{p}_V$ projects $\tau_V E$ onto the vertical space of order two $\tau_v(E_x)$, (ii) $\forall \lambda \in \mathbb{R}$, $\tilde{p}_{\tilde{\lambda} V} \circ d^2\tilde{\lambda}(V) = d^2\tilde{\lambda}(V) \circ \tilde{p}_V$, (iii) the restriction of $\tilde{p}$ to first order vectors is a connection p of order one on M.

Definition 2.2. Let $\tilde{p}$ be a connection of order two on M. For every continuous semimartingale (Y_t) in E and every second order form Θ on E, the integral $\int \Theta \, \mathbf{d}^{\mathcal{V}} Y = \int \tilde{p}^*(\Theta) \, \mathbf{d}Y$ is called the stochastic continuous covariant integral of Θ along (Y_t), where $\forall V \in E, \forall \mathbb{L} \in \tau_V E$, $\tilde{p}^*(\Theta)_V(\mathbb{L}) = \Theta_V \circ \tilde{p}_V(\mathbb{L})$.

To see Stratonovich and Itô integrals as second order integrals, one needs to specify how the connection p is transformed into connections $d_s p$ and Gp of order two.

2.1. The connection $d_s p$

Let p be a connection of order one on M. Then every $p^i : V \to p^i_V = dv^i + \Gamma^i_{jk}(x) v^k \, \mathrm{d}x^j : T_V E \to \mathbb{R}$ is a first order form on E, which can be transformed into a second order form on E by the operator d_s of Definition 1.1. In a way, the following definition extends this application d_s.

Definition 2.3. $d_s p$ is a map yielding for all V in E a smooth map $d_s p_V : \tau_V E \to \tau_v(E_x)$ defined by $\forall V \in E$, $\forall \mathbb{L} \in \tau_V E$, $d_s p_V(\mathbb{L}) = (d_s p^i)_V(\mathbb{L}) \frac{\partial}{\partial v^i} + p^i_V \cdot \mathrm{d}v^j(\mathbb{L}) \frac{\partial^2}{\partial v^i \partial v^j}$.

Note that we use coordinates $(V^i) = ((x^i, v^i)) = (x^i, dx^i(v))$ around $V = (x, v)$ on E, deduced from coordinates (x^i) around x on M. In [5], we show that $d_s p$ does not depend on the choice of these coordinates on M. For all V in E and $\mathbb{L}$ in $\tau_V E$, we can write the vector of order two $d_s p_V(\mathbb{L})$ as: $d_s p_V(\mathbb{L}) = [\mathrm{d}^2 v^i(\mathbb{L}) + \Gamma^i_{ml}(x) v^l \, \mathrm{d}^2 x^m(\mathbb{L}) + \frac{\partial \Gamma^i_{ml}}{\partial x^k}(x) v^l \, \mathrm{d}x^m \cdot \mathrm{d}x^k(\mathbb{L}) + \Gamma^i_{mk}(x) \, \mathrm{d}x^m \cdot \mathrm{d}v^k(\mathbb{L})] \frac{\partial}{\partial v^i} + [\mathrm{d}v^i \cdot \mathrm{d}v^j(\mathbb{L}) + \Gamma^i_{kl}(x) v^l \, \mathrm{d}x^k \cdot \mathrm{d}v^j(\mathbb{L})] \frac{\partial^2}{\partial v^i \partial v^j}$. Moreover, we get the following result.

Theorem 2.4. *For every connection p of order one on M, there exists a connection of order two, $d_s p$, such that the restriction of $d_s p$ to first order vectors is p, satisfying, for all first order form α on E: $d_s(p^*(\alpha)) = (d_s p)^*(d_s \alpha)$. As a consequence, the Stratonovich covariant integral along a continuous semimartingale can be expressed as a second order covariant integral*

$$\int \alpha \, \partial^{\mathcal{V}} Y = \int (d_s p)^*(d_s \alpha) \, \mathbf{d}Y.$$

In particular, we get for the covariant case an analogous formula than in Definition 1.1: $\int \alpha \, \partial^{\mathcal{V}} Y = \int d_s \alpha \, \mathbf{d}^{\mathcal{V}} Y$, *where* $\mathbf{d}^{\mathcal{V}}$ *is defined with the connection* $d_s p$.

2.2. The connection Gp

Let p be a connection of order one on M. Let τ be the associated parallel transport along geodesics on M. More precisely, for $V = (x, v)$ in E, consider the map $\tau_x(\cdot, \cdot) : E \to E_x$, such that, for all $W = (y, w)$ in E, $\tau_x(y, w)$ is the parallel transport of w along the geodesic (if it exists), joining y to x on M. The 2-jet of this map, at point V, is denoted by $(Gp)_V = \mathrm{d}^2 \tau_x(V) : \tau_V E \to \tau_v E_x$. In [5], we show that, for all V in E and $\mathbb{L}$ in $\tau_V E$, we can write the vector of order two $Gp_V(\mathbb{L})$ as: $Gp_V(\mathbb{L}) = [\mathrm{d}^2 v^i(\mathbb{L}) + \Gamma^i_{ml}(x) v^l \, \mathrm{d}^2 x^m(\mathbb{L}) + (\frac{\partial \Gamma^i_{lm}}{\partial x^k}(x) + \Gamma^i_{kj}(x) \Gamma^j_{lm}(x)) v^m \, \mathrm{d}x^l \cdot \mathrm{d}x^k(\mathbb{L}) + 2\Gamma^i_{mk}(x) \, \mathrm{d}x^m \cdot \mathrm{d}v^k(\mathbb{L})] \frac{\partial}{\partial v^i} + [\mathrm{d}v^i \cdot \mathrm{d}v^j(\mathbb{L}) + \Gamma^i_{kl}(x) v^l \Gamma^j_{mr}(x) v^r \mathrm{d}x^m \cdot \mathrm{d}x^k(\mathbb{L}) + 2\Gamma^i_{ml}(x) v^l \mathrm{d}x^m \cdot \mathrm{d}v^j(\mathbb{L})] \frac{\partial^2}{\partial v^i \partial v^j}$. Moreover we get the following result.

Theorem 2.5. *For every connection p of order one on M, there exists a connection of order two, Gp, such that its restriction to first order vectors is p, satisfying, for every first order form α on E, $\mathbb{G}(p^*(\alpha)) = (Gp)^*(\alpha)$ (where $\mathbb{G}$ is the map of Definition* 1.2 *considered on E, taken to be flat since E is a vector bundle). As a consequence, the Itô covariant integral along a continuous semimartingale can be expressed as a second order covariant integral*:

$$\int \alpha \, \mathrm{d}^{\mathcal{V}} Y = \int \tilde{p}^*(\alpha) \, \mathbf{d}Y.$$

In particular, we get an analogous formula to that in Definition 1.2: $\int \alpha \, \mathrm{d}^{\mathcal{V}} Y = \int \alpha \, \mathbf{d}^{\mathcal{V}} Y$, *where $\mathbf{d}^{\mathcal{V}}$ is defined with the connection $\tilde{p}$.*

3. Conclusion

We showed that covariant calculus on manifolds is a second order calculus. This second order formalism was used in the classical case by Cohen in [1] to extend stochastic calculus on manifolds to processes with jumps, in order to give discretization theorems for s.d.e. on manifolds. A function describes the jumps, its 2-jet permits to recover the continuous case. In the covariant framework, according to Definition 2.2, a covariant jump is described by a function whose 2-jet is a connection of order two. We saw a fundamental example with the parallel transport along geodesics. Using [1], [10] and [9], we have introduced in [5] a more general notion of transport, allowing us to define Itô and Stratonovich covariant integrals with jumps.

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