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Dynamical Systems

# Frequently dense orbits

K.-G. Grosse-Erdmann [a,1], Alfredo Peris [b,2]

[a] *Fachbereich Mathematik, FernUniversität Hagen, 58084 Hagen, Germany*
[b] *E.T.S. Arquitectura, Departament de Matemàtica Aplicada and IMPA-UPV, Universitat Politècnica de València, 46022 València, Spain*

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**Abstract**

We study the notion of frequent hypercyclicity that was recently introduced by Bayart and Grivaux. We show that frequently hypercyclic operators satisfy the Hypercyclicity Criterion, answering a question of Bayart and Grivaux [Trans. Amer. Math. Soc., in press]. We also disprove a conjecture therein concerning frequently hypercyclic weighted shifts, and we prove that vectors which have a somewhere frequently dense orbit are frequently hypercyclic. ***To cite this article: K.-G. Grosse-Erdmann, A. Peris, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***


**Résumé**

**Orbites fréquemment denses.** On étudie la notion d'hypercyclicité fréquente qui a récemment été introduite par Bayart et Grivaux. Nous démontrons que tout opérateur fréquemment hypercyclique vérifie le Critère d'Hypercyclicité, ce qui répond à une question de Bayart et Grivaux [Trans. Amer. Math. Soc., à paraître]. De plus, nous réfutons une conjecture de Bayart et Grivaux concernant les shifts à poids fréquemment hypercycliques, et nous démontrons que tout vecteur avec une orbite qui est quelque part fréquemment dense est fréquemment hypercyclique. ***Pour citer cet article : K.-G. Grosse-Erdmann, A. Peris, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***


*E-mail addresses:* kg.grosse-erdmann@fernuni-hagen.de (K.-G. Grosse-Erdmann), aperis@mat.upv.es (A. Peris).
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Un opérateur $T$ sur un espace vectoriel topologique séparable $X$ est dit *hypercyclique* s'il existe un vecteur $x \in X$ tel que l'orbite $\{T^n x\colon\ n = 1, 2, 3, \ldots\}$ est dense dans $X$. Bayart et Grivaux [1,2] ont récemment introduit la notion d'hypercyclicité fréquente en quantifiant la fréquence avec laquelle l'orbite rencontre tout ouvert non-vide.

**Définition 0.1.** Un opérateur $T$ sur un espace vectoriel topologique séparable $X$ est dit *fréquemment hypercyclique* s'il existe un vecteur $x \in X$ tel que, pour tout ouvert non-vide $U$ de $X$,

$$\underline{\mathrm{dens}}\{n \in \mathbb{N}\colon\ T^n x \in U\} > 0,$$

où, pour toute partie $A$ de $\mathbb{N}$, $\underline{\mathrm{dens}}(A) = \liminf_{N\to\infty} \frac{1}{N} \operatorname{card}\{n \leqslant N\colon\ n \in A\}$. Un tel vecteur est appelé *vecteur fréquemment hypercyclique* pour $T$.

Bayart et Grivaux [2, Question 4.8] demandent si, pour tout opérateur fréquemment hypercyclique $T$, l'opérateur $T \oplus T$ est hypercyclique. D'après un résultat de Bès et Peris [4], cette dernière propriété de $T$ est équivalente au Critère d'Hypercyclicité. Par conséquent, le théorème suivant implique une réponse positive.

**Théorème 0.2.** *Tout opérateur fréquemment hypercyclique sur un espace métrisable complet vérifie le Critère d'Hypercyclicité.*

Dans la Conjecture 2.10 de [2], Bayart et Grivaux proposent une condition pour caractériser l'hypercyclicité fréquente d'un shift à poids $T$ sur $\ell^p$, $1 \leqslant p < \infty$, donné par $T(x_n) = (w_{n+1}x_{n+1})$ avec des poids $(w_n)$ positifs. Nous donnons une condition nécessaire pour que $T$ soit fréquemment hypercyclique, et nous prouvons que cette condition est strictement plus forte que celle de la Conjecture 2.10, ce qui réfute la conjecture.

De plus, nous donnons l'analogue du résultat de Bourdon et Feldman [6] que toute orbite qui est quelque part dense est partout dense.

**Théorème 0.3.** *Soit $T$ un opérateur sur un espace vectoriel topologique $X$. S'il existe un vecteur $x \in X$ et un ouvert non-vide $U$ de $X$ tel que $\underline{\mathrm{dens}}\{n \in \mathbb{N}\colon\ T^n x \in V\} > 0$ pour toute partie ouverte non-vide $V$ de $U$, alors $x$ est fréquemment hypercyclique pour $T$.*

Comme corollaire on obtient l'analogue du résultat de Costakis [7] et Peris [9] que tout opérateur multi-hypercyclique est hypercyclique.

**Théorème 0.4.** *Soit $T$ un opérateur sur un espace vectoriel topologique $X$. S'il existe des vecteurs $x_1, \ldots, x_N \in X$ tels que, pour tout ouvert non-vide $U$ de $X$, $\underline{\mathrm{dens}} \bigcup_{j=1}^{N} \{n \in \mathbb{N}\colon\ T^n x_j \in U\} > 0$, alors un des vecteurs $x_j$, $j = 1, \ldots, N$, est fréquemment hypercyclique pour $T$.*

Ce dernier résultat implique que, pour tout opérateur fréquemment hypercyclique $T$ sur un espace vectoriel topologique, l'opérateur $T^N$, $N \in \mathbb{N}$, est aussi fréquemment hypercyclique, avec les mêmes vecteurs fréquemment hypercycliques que $T$, cf. [2, Theorem 4.7].

## 1. Introduction

A (continuous and linear) operator $T$ on a separable topological vector space $X$ is called *hypercyclic* if there exists a vector $x \in X$ whose orbit $\{T^n x\colon\ n = 1, 2, 3, \ldots\}$ is dense in $X$, that is, if its orbit meets every non-empty open subset $U$ of $X$. Motivated by Birkhoff's ergodic theorem, Bayart and Grivaux [1,2] have recently introduced the notion of a frequently hypercyclic operator by quantifying the frequency with which an orbit meets $U$. We recall that the *lower density* of a subset $A$ of $\mathbb{N}$ is defined as

$$\underline{\text{dens}}(A) = \liminf_{N\to\infty} \frac{\text{card}\{n \leqslant N\colon\ n \in A\}}{N}.$$

**Definition 1.1.** An operator $T$ on a separable topological vector space $X$ is called *frequently hypercyclic* if there exists a vector $x \in X$ such that, for every non-empty open subset $U$ of $X$, the set

$$\{n \in \mathbb{N}\colon\ T^n x \in U\}$$

has positive lower density. Each such vector $x$ is then called *frequently hypercyclic* for $T$.

This new concept has been thoroughly studied by Bayart and Grivaux [2] and Bonilla and the first author [5]. In this note we show that every frequently hypercyclic operator satisfies the Hypercyclicity Criterion, thereby solving Question 4.8 of [2], and we give a counterexample to Conjecture 2.10 of [2].

In our proofs we shall need the following well-known result, cf. [10]. The definition of *upper density* for subsets of $\mathbb{N}$ is analogous to that of lower density.

**Theorem 1.2.** *If $A \subset \mathbb{N}$ has positive upper density then the difference set $D := A - A = \{m - n\colon\ m, n \in A, m > n\}$ has bounded gaps, that is, there is some $M \in \mathbb{N}$ such that $D \cap [n, n + M] \neq \emptyset$ for all $n \in \mathbb{N}$.*

We begin, however, with a remarkable general property of frequent hypercyclicity. By a theorem of Bourdon and Feldman [6], an operator on a separable locally convex space is hypercyclic as soon as it has a somewhere dense orbit; in fact, the assumption of local convexity may be dropped, cf. Wengenroth [11]. We obtain the analogue for frequent hypercyclicity.

**Theorem 1.3.** *Let $T$ be an operator on a topological vector space $X$. If there is a vector $x \in X$ and a non-empty open subset $U$ of $X$ such that $\underline{\text{dens}}\{n \in \mathbb{N}\colon\ T^n x \in V\} > 0$ for all non-empty open subsets $V$ of $U$, then $x$ is frequently hypercyclic for $T$.*

**Proof.** It follows from the theorem of Bourdon and Feldman that $T$ is hypercyclic, hence topologically transitive. Thus, if $U'$ is an arbitrary non-empty open subset of $X$ there is some $N \in \mathbb{N}$ such that $T^N(U) \cap U' \neq \emptyset$. Hence there is a non-empty open subset $V$ of $U$ such that $T^N(V) \subset U'$, whence $\underline{\text{dens}}\{n \in \mathbb{N}\colon\ T^n x \in U'\} \geqslant \underline{\text{dens}}\{N + n\colon\ T^n x \in V\} > 0$, so that $x$ is frequently hypercyclic for $T$. □

As a consequence we obtain the analogue for frequent hypercyclicity of the result of Costakis [7] and the second author [9] that every multi-hypercyclic operator is hypercyclic. The proof is by induction on $N$.

**Theorem 1.4.** *Let $T$ be an operator on a topological vector space $X$. If there are vectors $x_1, \ldots, x_N \in X$ such that, for any non-empty open subset $U$ of $X$, $\underline{\text{dens}} \bigcup_{j=1}^{N} \{n \in \mathbb{N}\colon\ T^n x_j \in U\} > 0$, then some $x_j$, $j = 1, \ldots, N$, is frequently hypercyclic for $T$.*

This result, in turn, implies that for any frequently hypercyclic operator $T$ on a topological vector space $X$ the operator $T^N$, $N \in \mathbb{N}$, is also frequently hypercyclic, and it has the same frequently hypercyclic vectors as $T$. This generalizes Theorem 4.7 of [2].

## 2. The Hypercyclicity Criterion

In this section we let $X$ denote an F-space, that is, a complete and metrizable topological vector space, and we assume that $X$ is separable. In that case the so-called Hypercyclicity Criterion gives a sufficient condition for an operator on $X$ to be hypercyclic, see [4] or [8]. For the purpose of this note it will be enough to consider the criterion in one of its equivalent forms.

**Theorem 2.1** [3,4]. *Let $T$ be an operator on a separable F-space $X$. Then the following assertions are equivalent*:

(i) *$T$ satisfies the Hypercyclicity Criterion*;
(ii) *the operator $T \oplus T$ is hypercyclic*;
(iii) *for any non-empty open subsets $U$ and $V$ of $X$ and for any neighbourhood $W$ of $0$ in $X$ there exists an $n \in \mathbb{N}$ such that $T^n(U) \cap W \neq \emptyset$ and $T^n(W) \cap V \neq \emptyset$.*

One of the main problems in the theory of hypercyclic operators asks if every hypercyclic operator satisfies the Hypercyclicity Criterion. It seems to be accepted wisdom in hypercyclicity that the answer is positive if the operator has some additional regularity, like, for example, a dense set of periodic points (so that it becomes chaotic), see [4]. However, frequent hypercyclicity seems to impose rather more irregularity than regularity; hence the answer to Question 4.8 of [2] if every frequently hypercyclic operator satisfies the Hypercyclicity Criterion is not evident. In fact, the answer turns out to be in the affirmative.

**Theorem 2.2.** *Every frequently hypercyclic operator on a separable F-space satisfies the Hypercyclicity Criterion.*

**Proof.** Suppose that $T$ is frequently hypercyclic. We want to show that condition (iii) of Theorem 2.1 is satisfied. Thus, let $U$ and $V$ be non-empty open subsets of $X$ and $W$ a neighbourhood of 0 in $X$.

Since $T$ is necessarily hypercyclic, hence topologically transitive, there is some $N \in \mathbb{N}$ such that $T^N(U) \cap W \neq \emptyset$. Then there is a non-empty open subset $U_0$ of $U$ such that $T^N(U_0) \subset W$. Moreover, if $x$ is frequently hypercyclic for $T$ there is $A \subset \mathbb{N}$ of positive lower density such that $T^n x \in U_0$ for all $n \in A$, hence $T^{N+m-n} T^n x \in T^N(U_0) \subset W$ for all $m, n \in A$ with $m > n$. Letting $D = A - A$ we deduce that

$$T^k(U) \cap W \neq \emptyset \quad \text{for all } k \in N + D. \tag{1}$$

By Theorem 1.2 the set $N + D$ has bounded gaps, so that there is some $M \in \mathbb{N}$ with

$$(N + D) \cap [n, n + M] \neq \emptyset \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

Now, since $W$ is a neighbourhood of 0 there exists a 0-neighbourhood $W_0$ such that $T^n(W_0) \subset W$ for $n = 0, 1, \ldots, M$. Again by topological transitivity of $T$, there exists some $K > M$ and some $w_0 \in W_0$ with $T^K w_0 \in V$, hence $T^{K-n} T^n w_0 \in V$ for $n = 0, 1, \ldots, K$. Therefore we have

$$T^k(W) \cap V \neq \emptyset \quad \text{for all } k \in [K - M, K]. \tag{3}$$

Thus, (1), (2) and (3) imply that there exists some $k \in \mathbb{N}$ with $T^k(U) \cap W \neq \emptyset$ and $T^k(W) \cap V \neq \emptyset$, which had to be shown. □

We have thus shown that for any frequently hypercyclic operator $T$ the operator $T \oplus T$ is hypercyclic. Question 4.9 of [2] if $T \oplus T$ is even frequently hypercyclic, however, remains open.

## 3. Weighted backward shifts

An operator $T$ on the sequence space $\ell^p$, $1 \leqslant p < \infty$, is a weighted backward shift if there is a positive, and necessarily bounded, sequence $(w_n)$ such that $T(x_n) = (w_{n+1}x_{n+1})$ for $(x_n)_{n\geqslant 0} \in \ell^p$. In [2] it is shown that

$$\sum_{n=1}^{\infty} \frac{1}{(w_1 \cdots w_n)^p} < \infty \tag{4}$$

is a sufficient condition for frequent hypercyclicity of $T$, see also [5], while the following is obtained as a necessary condition: there exists a sequence $(n_k)$ of positive lower density such that

$$\sum_{k=1}^{\infty} \frac{1}{(w_1 \cdots w_{n_k})^p} < \infty. \tag{5}$$

In Conjecture 2.10 of [2] it is suggested that the latter condition, in fact, characterizes frequent hypercyclicity of $T$. We show here that this is not the case; to this end we derive a stronger necessary condition.

**Proposition 3.1.** *If a weighted backward shift $T$ with weights $(w_n)$ is frequently hypercyclic on $\ell^p$ then, for every $\varepsilon > 0$, there exists a sequence $(n_k)$ of positive lower density such that, for all $i \in \mathbb{N}$,*

$$\sum_{k=i+1}^{\infty} \frac{1}{(w_1 \cdots w_{n_k - n_i})^p} < \varepsilon.$$

**Proof.** Let $x \in \ell^p$ be frequently hypercyclic for $T$ and $0 < \eta < 1$. Then there exists a sequence $(n_k)$ of positive lower density such that $\|T^{n_k}x - e_0\| < \eta$ for all $k \in \mathbb{N}$, where $e_0 = (1, 0, 0, \ldots)$. This implies that

$$\big|(w_1 \cdots w_{n_k})x_{n_k} - 1\big| < \eta,$$

hence

$$(w_1 \cdots w_{n_k})|x_{n_k}| > 1 - \eta \quad \text{for all } k \in \mathbb{N}. \tag{6}$$

Moreover, we have for all $i \in \mathbb{N}$

$$\eta^p > \|T^{n_i}x - e_0\|^p \geqslant \sum_{j=1}^{\infty} \left(\frac{w_1 \cdots w_{j+n_i}}{w_1 \cdots w_j}\right)^p |x_{j+n_i}|^p \geqslant \sum_{k=i+1}^{\infty} \left(\frac{w_1 \cdots w_{n_k}}{w_1 \cdots w_{n_k - n_i}}\right)^p |x_{n_k}|^p,$$

hence by (6)

$$\sum_{k=i+1}^{\infty} \frac{1}{(w_1 \cdots w_{n_k - n_i})^p} < \frac{\eta^p}{(1-\eta)^p},$$

which implies the result. □

**Example 1.** There exists a weighted backward shift on $\ell^p$ with $\lim_{n\to\infty} w_n = 1$ that satisfies condition (5) for some sequence $(n_k)$ of positive lower density but that is not frequently hypercyclic.

**Proof.** Let $N_j = 2(j-1)j$ for $j \in \mathbb{N}$. We define $w_n = \frac{(n+1)^2}{n^2}$ for $N_j < n \leqslant N_j + j$, $w_n = (N_j + j + 1)^{-2/j}$ for $N_j + j < n \leqslant N_j + 2j$, $w_n = 1$ for $N_j + 2j < n \leqslant N_j + 3j$ and $w_n = (N_{j+1} + 1)^{2/j}$ for $N_j + 3j < n \leqslant N_j + 4j = N_{j+1}$, where $j \in \mathbb{N}$. Then $\lim_{n\to\infty} w_n = 1$; and we have $w_1 \cdots w_n = (n+1)^2$ for all $n \in \bigcup_{j\geqslant 1}[N_j + 1, N_j + j]$, so that condition (5) is satisfied for this set of positive lower density. Now, if $T$ were frequently hypercyclic then, by Proposition 3.1, there would exist a sequence $(m_k)$ of positive lower density such that $w_1 \cdots w_{m_k - m_i} > 1$ for all $k > i \geqslant 1$. On the other hand, we have $w_1 \cdots w_n = 1$ for $N_j + 2j < n \leqslant N_j + 3j$, hence for $j$ consecutive values of $n$, where $j$ is arbitrarily large. By Theorem 1.2 this is impossible. □

We do not know if the necessary condition of Proposition 3.1 is also sufficient for frequent hypercyclicity, nor if condition (4) is necessary.

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