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Characterization of the Wishart distributions on homogeneous cones

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Abstract

The aim of this Note is to give an extension to the Wishart distribution on homogeneous cones of the characterization of the ordinary Wishart on symmetric matrices, as given by Bobecka and Wesołowski [Studia Math. 152 (2002) 147–160]. Our method of proof is parallel to theirs. We also define the beta distribution on homogeneous cones, which appears in the course of this characterization. ***To cite this article: I. Boutouria, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Caractérisation des lois de Wishart sur les cônes homogènes. Le résultat principal de cette Note est la caractérisation de la loi de Wishart sur le cône homogène d'une algèbre de Vinberg. Ce résultat est obtenu par une méthode parallèle à celle utilisé par Bobecka et Wesołowski [Studia Math. 152 (2002) 147–160] et représente une extention de leur résultat concernant les lois de Wishart ordinaires sur les matrices symétriques. Nous définissons aussi la loi bêta sur les cones homogènes qui apparait dans cette caracterisation. ***Pour citer cet article : I. Boutouria, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Andersson et Wojnar [1] ont introduit une classe de lois de Wishart généralisées sur le cône homogène d'une algèbre de Vinberg. Ces lois représentent une généralisation des lois de Wishart ordinaires sur le cône des matrices définies positives réelles ou plus généralement sur le cône symétrique d'une algèbre de Jordan quelconque (voir [3]) ; elles servent en inférence statistique dans le cas où la matrice de covariance est un paramètre défini par des relations de symétrie ou des relations d'indépendance statistique des variables gaussiennes originales.

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Soit $\mathcal{A} = \prod_{i,j \in I \times I} \mathcal{A}_{ij}$ où les $\mathcal{A}_{ij}$ sont définis plus loin en (1). Les éléments de $\mathcal{A}$ peuvent être considérés comme des matrices dont les entrées sont des éléments d'espaces vectoriels. Les sous espaces de $\mathcal{A}$ des matrices triangulaires supérieures et triangulaires inférieures définis par (2) et (3) sont respectivement notés $\mathcal{T}_u$ et $\mathcal{T}_l$. Les ensembles des matrices triangulaires supérieures et triangulaires inférieures dont les éléments diagonaux sont positifs sont notés $\mathcal{T}_u^+$ et $\mathcal{T}_l^+$. Les ensembles des matrices triangulaires supérieures et triangulaires inférieures dont les éléments diagonaux sont égaux à 1 sont notés $\mathcal{T}_u^1$ et $\mathcal{T}_l^1$. On désigne par $\mathcal{D}$ l'ensemble des matrices diagonales et on pose $\mathcal{D}^+ = \mathcal{D} \cap \mathcal{T}_u^+$. Aussi, on désigne par $\operatorname{tr} A = \sum_{i \in I} a_{ii}$ la trace de $A \equiv (a_{ij},\ i, j \in I)$. On prend l'élément de $\mathcal{D}$ tel que $a_{ii} = 1,\ \forall i \in I$ comme élément unité de $\mathcal{A}$, on le note par e.

On introduit des applications bilinéaires $\mathcal{A}_{ij} \times \mathcal{A}_{jk} \to \mathcal{A}_{ik}$; $(a_{ij}, b_{jk}) \mapsto a_{ij} b_{jk}$, pour i, j, k dans I et on définit sur $\mathcal{A}$ une multiplication $(A, B) \mapsto AB$ par $AB = C \equiv (c_{ij},\ i, j \in I)$ avec $c_{ij} = \sum_{\mu \in I} a_{i\mu} b_{\mu j}$. Si ce produit satisfait les propriétés (i)–(vi) de (4), alors $\mathcal{A}$ est dite une algèbre de Vinberg.

Vinberg [5] a montré que $\mathcal{P} = \{TT^* \in \mathcal{A},\ T \in \mathcal{T}_l^+\} = \{TDT^* \in \mathcal{A},\ T \in T_l^1, D \in \mathcal{D}^+\}$ est un cône homogène. Alors si $S = TDT^* \equiv (s_{ij},\ i, j \in I)$ avec $T \in T_l^1$ et $D \in \mathcal{D}^+$, on note $s_{[i].} = D_{ii}$.

Soit G la composante connexe de l'identité dans $\operatorname{Aut}(\mathcal{P})$ et soit $\pi : T \in \mathcal{T}_l^+ \mapsto \pi(T) \in G$ telle que pour $X = VV^* \in \mathcal{P}$, avec $V \in \mathcal{T}_l^+$, $\pi(T)(X) = (TV)(V^*T^*)$.

Soit $\mathcal{X}$ l'ensemble des multiplicateurs de G tels que les $(\lambda_i,\ i \in I)$ correspondant à $\chi \circ \pi$ vérifient $\lambda_i > \frac{n_{i.}}{2}$, $i \in I$. La loi de Wishart sur $\mathcal{P}$ de paramètre $\sigma \in \mathcal{P}$ et de multiplicateur $\chi \in \mathcal{X}$ est alors définie par

$$HW_{\chi,\sigma}(\mathrm{d}x) = \frac{\pi^{(|I| - n_.)/2} \prod_{i \in I} \lambda_i^{\lambda_i} \prod_{i \in I} x_{[i].}^{\lambda_i - n_i}}{\prod_{i \in I} \Gamma(\lambda_i - n_{i.}/2) \prod_{i \in I} \sigma_{[i].}^{\lambda_i}} \, \mathrm{e}^{-\{\operatorname{tr}(\sigma^{-\chi} x)\}} \mathbf{1}_{\mathcal{P}}(x)\, \mathrm{d}x.$$

Pour plus de détails, on peut se référer à [1].

Pour la caractérisation de cette loi de Wishart, nous définissons sur $\mathcal{P}$ l' algorithme de division suivant $g : \mathcal{P} \to G$; $S = TT^* \mapsto g(S) = \pi(T^{-1})$. On peut dire que $g(S)X$ représente le « quotient » de X par S.

Le résultat principal de cette note est donné dans le théorème suivant.

Théorème 0.1. *Soient X et Y deux variables aléatoires indépendantes à valeurs dans $\mathcal{P}$ de densités deux fois différentiables. Soient $S = X + Y$ et $U = g(S)(X)$. Si $U = g(S)(X)$ est indépendante de S, alors il existe $\chi, \chi' \in \mathcal{X}$ et $\sigma \in \mathcal{P}$ tels que $X \sim HW_{\chi,\sigma}$ et $Y \sim HW_{\chi',\sigma}$.*

Au cours de cette caractérisation, apparait une loi bêta qui est indépendante de l'algorithme de division choisi et qui est donnée dans la proposition suivante.

Proposition 0.2. *Soit $\mathcal{P}$ un cône homogène et soient X et X' deux variables aléatoires indépendantes de lois $HW_{\chi,\sigma}$ et $HW_{\chi',\sigma}$ de paramètre $\sigma \in \mathcal{P}$ et de multiplicateurs $\chi = \{\lambda_i,\ i \in I\}$ et $\chi' = \{\lambda'_i,\ i \in I\}$ respectivement. Alors la loi de la variable aléatoire $U = g(X + X')X$ est donnée par*

$$H\beta_{\chi,\chi'}(\mathrm{d}u) = \frac{\pi^{(|I| - n_.)/2} \prod_{i \in I} \Gamma(\lambda_i + \lambda'_i - n_{i.}/2)}{\prod_{i \in I} \Gamma(\lambda_i - n_{i.}/2) \Gamma(\lambda'_i - n_{i.}/2)} \times \prod_{i \in I} u_{[i].}^{\lambda_i - n_i} (e - u)_{[i].}^{\lambda'_i - n_i} \mathbf{1}_{\mathcal{P} \cap (e - \mathcal{P})}(u)\, \mathrm{d}u.$$

1. Introduction

Andersson and Wojnar [1] have introduced a class of generalized Wishart distributions on the homogeneous cone of a Vinberg algebra. These distributions represent a generalization of the ordinary Wishart distributions on the cone of real positive definite matrices or on any symmetric cone (see [3]). They arise in statistical inference for Gaussian models with covariance matrices structured by relations of symmetry or dependence between the variables.

Let I be a partially ordered finite set equipped with a relation denoted $\preceq$. We will write $i \prec j$ if $i \preceq j$ and $i \neq j$. We assume that I satisfies the following condition: for any two points i and j in I such that either $i \prec j$ or $j \prec i$ the path on the Hasse diagram of I between i and j is unique. For all pairs $(i,j) \in I \times I$ with $j \prec i$, let E_{ij} be a finite-dimensional vector space over $\mathbb{R}$ with $n_{ij} = \dim(E_{ij}) > 0$. Set

$$\mathcal{A}_{ij} = \begin{cases} \mathbb{R} & \text{for } i = j, \\ E_{ij} & \text{for } j \succ i \text{ or } j \prec i, \\ \{0\} & \text{otherwise} \end{cases} \tag{1}$$

and $\mathcal{A} = \prod_{i,j \in I \times I} \mathcal{A}_{ij}$. Define $n_{i.} = \sum_{\mu < i} n_{i\mu}$, $n_{.i} = \sum_{i<\mu} n_{\mu i}$, $n_i = 1 + \frac{1}{2}(n_{i.} + n_{.i})$, $i \in I$ and $n_{.} = \sum_{i \in I} n_i$.

An element $A \equiv (a_{ij},\ i,j \in I)$ of $\mathcal{A}$ may be seen as a matrix and so we define the trace $\operatorname{tr} A = \sum_{i \in I} a_{ii}$.

Let $f_{ij} : E_{ij} \to E_{ij}, i \succ j$, be involutional linear mappings, i.e., $f_{ij}^{-1} = f_{ij}$. They induce an involutional mapping $(A \mapsto A^*)$ of $\mathcal{A}$ given as follows: $A^* = (a^*_{ij} \mid (i,j) \in I \times I)$, where

$$a^*_{ij} = \begin{cases} a_{ii} & \text{for } i = j, \\ f_{ij}(a_{ij}) = a^*_{ij} & \text{for } j \prec i \text{ or } i \prec j, \\ \{0\} & \text{otherwise.} \end{cases}$$

We now define the following subspaces of $\mathcal{A}$:
the upper triangular matrices

$$\mathcal{T}_u = \{A \equiv (a_{ij}) \in \mathcal{A},\ \forall i,j \in I\colon\ i \npreceq j \Rightarrow a_{ij} = 0\}; \tag{2}$$

the lower triangular matrices

$$\mathcal{T}_l = \{A \equiv (a_{ij}) \in \mathcal{A},\ \forall i,j \in I\colon\ j \npreceq i \Rightarrow a_{ij} = 0\}; \tag{3}$$

and the Hermitian matrices $\mathcal{H} = \{A \in \mathcal{A},\ A^* = A\}$.

The sets of upper and lower triangular matrices in $\mathcal{P}$ with positive diagonal elements are respectively denoted by $\mathcal{T}_u^+$ and $\mathcal{T}_l^+$. The sets of upper and lower triangular matrices with all diagonal elements equal to 1 are respectively denoted by $\mathcal{T}_u^1$ and $\mathcal{T}_l^1$. The sets of diagonal matrices and of diagonal matrices with positive entries are denoted by $\mathcal{D}$ and $\mathcal{D}^+$, respectively.

We are going to equip the vector space $\mathcal{A}$ with a bilinear map called multiplication and denoted by $(A,B) \mapsto AB$. For this purpose we need to define bilinear mappings $\mathcal{A}_{ij} \times \mathcal{A}_{jk} \to \mathcal{A}_{ik}$, denoted by $(a_{ij}, b_{jk}) \mapsto a_{ij} b_{jk}$, and then define $AB = C \equiv (c_{ij} \mid (i,j) \in I \times I)$ by $c_{ij} = \sum_{\mu \in I} a_{i\mu} b_{\mu j}$.

The multiplication is required to satisfy the following properties:

$$\begin{array}{l|l} \text{(i) } \forall A \in \mathcal{A};\ A \neq 0 \Rightarrow \operatorname{tr}(AA^*) > 0 & \text{(iv) } \forall A,B,C \in \mathcal{A};\ \operatorname{tr}(A(BC)) = \operatorname{tr}((AB)C) \\ \text{(ii) } \forall A,B \in \mathcal{A};\ (AB)^* = B^*A^* & \text{(v) } \forall U,S,T \in \mathcal{T}_l;\ (ST)U = S(TU) \\ \text{(iii) } \forall A,B \in \mathcal{A};\ \operatorname{tr}(AB) = \operatorname{tr}(BA) & \text{(vi) } \forall U,S,T \in \mathcal{T}_l;\ T(UU^*) = (TU)U^*. \end{array} \tag{4}$$

An algebra $\mathcal{A}$ with the above structure and properties is called a Vinberg algebra (For more details, we can refer to [1].) We choose the element $A \equiv (a_{ij} \mid (i,j) \in I \times I)$ of $\mathcal{D}$ such that $a_{ii} = 1$, $\forall i \in I$ as the unit element of $\mathcal{A}$ and we denote it by e. Vinberg [5] proved that the subset $\mathcal{P} = \{TT^* \in \mathcal{A},\ T \in \mathcal{T}_l^+\} \subset \mathcal{H} \subset \mathcal{A}$ forms a homogeneous cone, that is the action of its automorphism group is transitive.

The definition of $\mathcal{P}$ could be changed to the following equivalent definition $\mathcal{P} = \{TDT^* \in \mathcal{A},\ T \in \mathcal{T}_l^1,\ D \in \mathcal{D}^+\}$. The two decompositions $S = TT^*$, $T \in \mathcal{T}_l^+$ and $S = T_1 D T_1^*$, $T_1 \in \mathcal{T}_l^1$, $D \in \mathcal{D}^+$ are unique and their connection is given by $T = T_1\sqrt{D}$ where $\sqrt{D} \equiv \operatorname{diag}(\sqrt{d_i},\ i \in I) \in \mathcal{D}^+$ with $D \equiv \operatorname{diag}(d_i,\ i \in I) \in \mathcal{D}^+$.

For $S = (s_{ij},\ i,j \in I) = T_1 D T_1^*$, we write $D_{ii} = s_{[i].}$ (see [4]).

Let G be the connected component of the identity in $\operatorname{Aut}(\mathcal{P})$. Let $\chi : G \to \mathbb{R}_+$ be a multiplier on G and consider the map $\pi : T \in \mathcal{T}_l^+ \mapsto \pi(T) \in \pi(\mathcal{T}_l^+) \subset G$ such that for $X = VV^* \in \mathcal{P}$, $V \in \mathcal{T}_l^+$ $\pi(T)(X) = (TV)(V^*T^*)$.

In [1], it has been shown that the restriction of a multiplier χ to the (lower) triangular group $\mathcal{T}_l^+$, i.e., $\chi \circ \pi : T_l^+ \to \mathbb{R}_+$ is in one to one correspondence with the set of $(\lambda_i, i \in I) \in \mathbb{R}^I$. To each χ corresponds a unique (up to a multiplicative constant) equivariant measure ν^χ. For a given χ, the family of Wishart distributions on $\mathcal{P}$ is defined as the natural exponential family generated by ν^χ.

Let $\mathcal{X} = \{\chi \circ \pi : T_l^+ \to \mathbb{R}_+;\ \lambda_i > \frac{n_{i.}}{2},\ i \in I\}$. The generalized Wishart distribution on $\mathcal{P}$ with parameters $\sigma \in \mathcal{P}$ and multiplier $\chi \in \mathcal{X}$ is defined in [1] by

$$HW_{\chi,\sigma}(\mathrm{d}x) = \frac{\pi^{(|I|-n_{.})/2} \prod_{i\in I} \lambda_i^{\lambda_i} \prod_{i\in I} x_{[i].}^{\lambda_i - n_i}}{\prod_{i\in I} \Gamma(\lambda_i - n_{i.}/2) \prod_{i\in I} \sigma_{[i].}^{\lambda_i}} \mathrm{e}^{-\{\mathrm{tr}(\sigma^{-\chi} x)\}} \mathbf{1}_{\mathcal{P}}(x)\,\mathrm{d}x.$$

2. Characterization of the Wishart distributions on a homogeneous cone

In this section we state our main result which is an extension to the Wishart on homogeneous cones of the characterization of the ordinary Wishart on symmetric matrices, as given by Bobecka and Wesołowski [2]. In order to state our characterization, we need to introduce the concept of a division algorithm.

Let $\mathcal{P}$ be an homogeneous cone. The measurable map

$$g : S = TT^* \in \mathcal{P} \mapsto g(S) = \pi(T^{-1}) \in G, \tag{5}$$

is a division algorithm. In fact $g(S)X$ can be viewed as the ratio of X by S.

Theorem 2.1. *Let $g : b \mapsto g(b)$ be the division algorithm defined by* (5)*. Let X and Y be independent random variables taking their values in $\mathcal{P}$, with strictly twice differentiable densities. Set $S = X + Y$ and $U = g(S)(X)$. If S and U are independent, then there exist χ, $\chi' \in \mathcal{X}$ and $\sigma \in \mathcal{P}$ such that $X \sim HW_{\chi,\sigma}$ and $Y \sim HW_{\chi',\sigma}$.*

The proof of Theorem 2.1 relies on the following two lemmas.

Lemma 2.2. *Let $a : \mathcal{P} \cap (e - \mathcal{P}) \to \mathbb{R}$ and $f : \mathcal{P} \to \mathbb{R}$ be functions such that, for any $u \in \mathcal{P} \cap (e - \mathcal{P})$ and $s \in \mathcal{P}$, $a(u) = f(g^{-1}(s)(u)) - f(g^{-1}(s)(e - u))$.*

Assume that f is differentiable then there exist real positive numbers β_i, $i \in I$, and $\lambda \in \mathbb{R}$ such that, for any $u \in \mathcal{P} \cap (e - \mathcal{P})$ and $s \in \mathcal{P}$,

$$a(u) = \sum_{i\in I} \beta_i \ln u_{[i].} (e-u)_{[i].}^{-1}, \qquad f(s) = \sum_{i\in I} \beta_i \ln s_{[i].} + \lambda.$$

Lemma 2.3. *Let $a_1 : \mathcal{P} \cap (e - \mathcal{P}) \to \mathbb{R}$, $a_2 : \mathcal{P} \to \mathbb{R}$ and $p : \mathcal{P} \to \mathbb{R}$ be functions satisfying*

$$a_1(u) + a_2(s) = p\big(g^{-1}(s)(u)\big) + p\big(g^{-1}(s)(e-u)\big), \tag{6}$$

for any $u \in \mathcal{P} \cap (e - \mathcal{P})$ and $s \in \mathcal{P}$. Assume that p is twice differentiable then there exist $\beta_i' \in \mathbb{R}$, for $i \in I$, $\delta \in \mathcal{H}$ and $c_1, c_2,\ c \in \mathbb{R}$ such that for any $u \in \mathcal{P} \cap (e - \mathcal{P})$ and $s \in \mathcal{P}$ $a_1(u) = \sum_{i\in I} \beta_i' \ln u_{[i].} (e-u)_{[i].} + c_1$, $a_2(s) = 2\sum_{i\in I} \beta_i' \ln s_{[i].} + \langle \delta, s\rangle + c_2$, $p(s) = \sum_{i\in I} \beta_i' \ln s_{[i].} + \langle \delta, s\rangle + c$.

Elements of proof of Lemma 2.3. Define $\varphi(t) = tt^*$, for any $t \in \mathcal{T}_l^+$.

Replacing s by $\varphi(t)$ and u by $\varphi(w)$ in (6) gives $a_1(\varphi(w)) + a_2(\varphi(t)) = p((tw)(w^*t^*)) + p(t(e - ww^*)t^*)$. Define $b_1 = a_1 \circ \varphi$ and $b_2 = a_2 \circ \varphi$. Then $b_1(w) + b_2(t) = p((tw)(w^*t^*)) + p(t(e - ww^*)t^*)$.

Differentiating this twice with respect to w gives

$$\begin{aligned}b_1''(w)(h)(k) = {} & \big(p''\big((tw)(w^*t^*)\big) + p''\big(t(e-ww^*)t^*\big)\big)\big((tk)(w^*t^*) + (tw)(k^*t^*)\big)\\ & \times \big((th)(w^*t^*) + (tw)(h^*t^*)\big)\\ & + \big(p'\big((tw)(w^*t^*)\big) - p'\big(t(e-ww^*)t^*\big)\big)\big((th)(k^*t^*) + (tk)(h^*t^*)\big),\end{aligned}$$

for any $h, k \in \mathcal{T}_l$.

Setting $w = \frac{e}{\sqrt{2}}$ and changing t by $\sqrt{2}t$ imply for any $h, k \in \mathcal{T}_l$,

$$b_1''\left(\frac{e}{\sqrt{2}}\right)(h)(k) = 8p''(tt^*)\big((tk)t^* + t(k^*t^*)\big)\big((th)t^* + t(h^*t^*)\big). \tag{7}$$

Let $\rho(t) = p \circ \varphi(t)$. Then for $h_0 \in \mathcal{T}_l$,

$$\rho'(t)(h_0) = p'\big(\varphi(t)\big)\varphi'(t)h_0. \tag{8}$$

Differentiating (8) once again with respect t gives

$$\rho''(t)(h_0)(k_0) = p''(tt^*)(tk_0^* + k_0t^*)(th_0^* + h_0t^*) + p'(tt^*)(k_0h_0^* + h_0k_0^*), \tag{9}$$

for any $h_0, k_0 \in \mathcal{T}_l$.

Let $h = t^{-1}h_0$ and $k = t^{-1}k_0$ in (7) and use (9) to obtain $b_1''(\frac{e}{\sqrt{2}})(t^{-1}h_0)(t^{-1}k_0) = 8(\rho''(t)(h_0)(k_0) - p'(tt^*)(h_0k_0^* + k_0h_0^*))$. This for $h_0 = t$ together with (8), leads to the differential equation

$$\rho''(t)t - \rho'(t) = \frac{1}{8}b_1''\left(\frac{e}{\sqrt{2}}\right)(e)t^{-1}. \tag{10}$$

Setting $\frac{1}{8}b_1''(\frac{e}{\sqrt{2}})(e) = (b_{ij})_{i,j\in I} \in \mathcal{T}_l$, we get

$$\rho''(t)t - \rho'(t) = \sum_{i\in I} b_{ii}t_{ii}^{-1}. \tag{11}$$

The general solution of $\rho''(t)t - \rho'(t) = 0$ is $\rho(t) = \langle\delta, tt^*\rangle + c$, where $\delta \in \mathcal{H}$ and $c \in \mathbb{R}$.

Since $\rho_0(t) = -\sum_{i\in I} \frac{b_{ii}}{2}\ln t_{ii}$, is a particular solution of (10), the solution of (11) is $\rho(t) = -\sum_{i\in I} \frac{b_{ii}}{2}\ln t_{ii} + \langle\delta, tt^*\rangle + c$, where $\delta \in \mathcal{H}$, $b_{ii} \in \mathbb{R}$ and $c \in \mathbb{R}$.

Hence $p(s) = \sum_{i\in I} \beta_i' \ln s_{[i].} + \langle\delta, s\rangle + c$, where $-\frac{b_{ii}}{4} = \beta_i'$. Conversely, if we take $p(s) = \sum_{i\in I} \beta_i' \ln s_{[i].} + \langle\delta, s\rangle + c$, then we can write $p(g^{-1}(s)(u)) + p(g^{-1}(s)(e-u)) = a_1(u) + a_2(s)$, where $a_1(u) = \sum_{i\in I} \beta_i' \ln u_{[i].}(e-u)_{[i].} + c_1$, $a_2(s) = 2\sum_{i\in I} \beta_i' \ln s_{[i].} + \langle\delta, s\rangle + c_2$. □

Elements of proof of Theorem 2.1. Since X and Y are independent and so are U and S, the following identity holds for all $u \in \mathcal{P} \cap (e-\mathcal{P})$ and $s \in \mathcal{P}$, $f_U(u)f_S(s) = \det g(s) f_X(g^{-1}(s)(u)) f_Y(g^{-1}(s)(e-u))$, where f_U, f_S, f_X and f_Y denote the densities of U, S, X and Y, respectively.

Then $f_1(u) + f_2(s) = f_3(g^{-1}(s)(u)) + f_4(g^{-1}(s)(e-u))$ where $f_1(u) = \ln f_U(u)$, $f_2(s) = \ln f_S(s) - \ln\det g(s)$, $f_3(x) = \ln f_X(x)$ and $f_4(y) = \ln f_Y(y)$.

It is easy to see that $f_1(u) - f_1(e-u) = f_3(g^{-1}(s)(u)) - f_4(g^{-1}(s)(u)) - [f_3(g^{-1}(s)(e-u)) - f_4(g^{-1}(s)(e-u))]$.

Now, according to Lemma 2.2, we obtain $a(u) = \sum_{i\in I} \beta_i \ln u_{[i].}(e-u)_{[i].}^{-1}$, $f(s) = \sum_{i\in I} \beta_i \ln s_{[i].} + \lambda$, for some β_i and λ in $\mathbb{R}$. Therefore $a_1(u) + a_2(s) = f_4(g^{-1}(s)(u)) + f_4(g^{-1}(s)(e-u))$, where $a_1(u) = f_1(u)$, $a_2(s) = f_2(s) - \sum_{i\in I} \beta_i \ln s_{[i].} - \lambda\cdot$.

Then by Lemma 2.3, we get $f_4(s) = \sum_{i\in I} \beta_i' \ln s_{[i].} + \langle\delta, s\rangle + \alpha$, for some $\beta_i' \in \mathbb{R}$, $\delta \in \mathcal{H}$ and $\alpha \in \mathbb{R}$. It follows that $f_Y(s) = \prod_{i\in I} s_{[i].}^{\beta_i'} \exp\langle\delta, s\rangle \exp\alpha\cdot$. Since f_Y is a probability density, we have that $\beta_i' > -1 - \frac{n_{.i}}{2}$, $i \in I$, and $-\delta = \sigma^{-\chi'} \in \mathcal{P}$, with $\sigma \in \mathcal{P}$ and $\chi' = \{\lambda_i = \beta_i' + n_i\} \in \mathcal{X}$. Hence $Y \sim HW_{\chi',\sigma}$.

Proceeding similarly, we also get $f_X(s) = \prod_{i\in I} s_{[i].}^{\beta_i+\beta_i'} \exp\langle\delta, s\rangle \exp(\alpha+\lambda)$, which implies that $\beta_i + \beta_i' > -1 - \frac{n_{.i}}{2}$, for all $i \in I$ and consequently $X \sim HW_{\chi,\sigma}$. $\square$

In the course of our characterization above, we have introduced the random variable $U = g(S)X$. If X and Y follow Wishart distributions with the same scale parameter σ, the distribution of this random variable is clearly the analog of the beta distribution for real symmetric matrices. We will call it the beta distribution on homogeneous cones. We have the following result:

Proposition 2.4. *Let $\mathcal{P}$ be an homogeneous cone and let X and X' be independent random variables with Wishart distributions on $\mathcal{P}$, $HW_{\chi,\sigma}$ and $HW_{\chi',\sigma}$ with parameter $\sigma \in \mathcal{P}$ and multipliers $\chi = \{\lambda_i, i \in I\}$ and $\chi' = \{\lambda_i', i \in I\}$ respectively. Then the random variable $U = g(X+X')X$ is independent of $X+X'$, its distribution does not depend upon the particular division algorithm and is equal to*

$$H\beta_{\chi,\chi'}(du) = \frac{\pi^{(|I|-n_{.})/2}\prod_{i\in I}\Gamma(\lambda_i+\lambda_i'-n_{i.}/2)}{\prod_{i\in I}\Gamma(\lambda_i-n_{i.}/2)\Gamma(\lambda_i'-n_{i.}/2)} \times \prod_{i\in I} u_{[i].}^{\lambda_i-n_i}(e-u)_{[i].}^{\lambda_i'-n_i}\mathbf{1}_{\mathcal{P}\cap(e-\mathcal{P})}(u)\,\mathrm{d}u. \tag{12}$$

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