Available online at www.sciencedirect.com
SCIENCE DIRECT
C. R. Acad. Sci. Paris, Ser. I 340 (2005) 755–758
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Probability Theory/Statistics

Archimax copulas and invariance under transformations ☆

Erich Peter Klement [a], Radko Mesiar [b,c], Endre Pap [d]

[a] *Department of Knowledge-Based Mathematical Systems, Johannes Kepler University, A-4040 Linz, Austria*
[b] *Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Bratislava, Slovakia*
[c] *Institute of the Theory of Information and Automation, Czech Academy of Sciences, Prague, Czech Republic*
[d] *Department of Mathematics and Informatics, University of Novi Sad, Novi Sad, Serbia and Montenegro*

Received 16 February 2005; accepted after revision 5 April 2005

Presented by Paul Deheuvels

Abstract

Copulas which are invariant under transformations by means of increasing bijections on the unit interval are investigated, and the relationship to maximum attractors and Archimax copulas is discussed. ***To cite this article: E.P. Klement et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Les copules Archimax et leur invariance par rapport aux transformations. On étudie les copules qui sont invariantes par rapport aux transformations par les bijections croissantes de l'intervalle unité, et on examine la relation entre les attracteurs des valeurs maximales et les copules Archimax. ***Pour citer cet article : E.P. Klement et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Introduction

Sklar's Theorem [10,12,13] states that each random vector (X, Y) is characterized by some copula C in the sense that for its joint distribution H_{XY} and for the corresponding marginal distributions F_X and F_Y we have $H_{XY}(x, y) = C(F_X(x), F_Y(y))$.

☆ Research supported by two European actions (CEEPUS network SK-42 and COST action 274) and by grants VEGA 1/1145/04, GACR 402/04/1026, and MNTRS-1866 and by the Academy of Sciences and Arts of Vojvodina.

E-mail addresses: ep.klement@jku.at (E.P. Klement), mesiar@math.sk (R. Mesiar), pap@im.ns.ac.yu, pape@eunet.yu (E. Pap).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2005.04.012

In this Note we investigate transformations of copulas by functions in one variable. Such transformations play a role in statistics: as an example, if $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ are iid random vectors (characterized by some copula C) then the random vector $(\max(X_1, X_2, \ldots, X_n), \max(Y_1, Y_2, \ldots, Y_n))$ is characterized by the $\varphi_{1/n}$-transform of C in the sense of (2) below with $\varphi_{1/n}(x) = x^{1/n}$ [14].

Recall that a (*two-dimensional*) *copula* is a function $C\colon [0,1]^2 \to [0,1]$ such that $C(0,x) = C(x,0) = 0$ and $C(1,x) = C(x,1) = x$ for all $x \in [0,1]$, and C is 2-increasing, i.e., for all $x, x^*, y, y^* \in [0,1]$ with $x \leqslant x^*$ and $y \leqslant y^*$ for the volume Vol_C of the rectangle $[x, x^*] \times [y, y^*]$ we have

$$\mathrm{Vol}_C\big([x,x^*]\times[y,y^*]\big) = C(x,y) - C(x,y^*) + C(x^*,y^*) - C(x^*,y) \geqslant 0. \tag{1}$$

Important examples of copulas are the *Fréchet–Hoeffding bounds* M and W given by $M(x,y) = \min(x,y)$ and $W(x,y) = \max(x+y-1, 0)$, respectively, and the *product* Π given by $\Pi(x,y) = x \cdot y$. Obviously, each copula C satisfies $W \leqslant C \leqslant M$.

2. Transformations of copulas

If Φ denotes the set of all increasing bijections from $[0,1]$ to $[0,1]$, then for each $\varphi \in \Phi$ and for each copula C consider the function $C_\varphi : [0,1]^2 \to [0,1]$ given by

$$C_\varphi(x,y) = \varphi^{-1}\big(C\big(\varphi(x), \varphi(y)\big)\big). \tag{2}$$

In general, C_φ is not necessarily a copula: consider $\varphi \in \Phi$ defined by $\varphi(x) = x^2$ then W_φ is not Lipschitz (see [8, Example 1.26]) and, therefore, not a copula. Evidently, $D \leqslant C$ implies $D_\varphi \leqslant C_\varphi$. Moreover, for all $\varphi, \xi \in \Phi$ we always get $(C_\varphi)_\xi = C_{\varphi\circ\xi}$. The transition from C to C_φ preserves many algebraic properties, among them commutativity and associativity as well as the existence of zero divisors and of idempotent elements.

If, for a copula C and some $\varphi \in \Phi$, we have $C_\varphi = C$ then C is called φ-*invariant*. As an immediate consequence, each φ-invariant copula is $\varphi_{(n)}$-invariant for each $n \in \mathbb{Z}$, where $\varphi_{(0)} = \mathrm{id}_{[0,1]}$ and, for each $n \in \mathbb{N}$, $\varphi_{(n)} = \varphi \circ \varphi_{(n-1)}$ and $\varphi_{(-n)} = (\varphi_{(n)})^{-1}$. Also, if C is both φ-invariant and ξ-invariant then it is $(\varphi \circ \xi)$-invariant. Moreover, C is φ-invariant if and only if C_ξ is $(\xi^{-1} \circ \varphi \circ \xi)$-invariant for each $\xi \in \Phi$. The only copula which is φ-invariant for all $\varphi \in \Phi$ is the minimum M.

If, for a copula C and some $\varphi \in \Phi$, the limit $\lim_{n\to\infty} C_{\varphi_{(n)}}$ exists and is a copula, then $C^*_\varphi = \lim_{n\to\infty} C_{\varphi_{(n)}}$ is called a φ-*attractor* of C. It is immediately seen that a copula D is a φ-attractor of some copula C if and only if D is φ-invariant. Also, if C^*_φ is a copula and $C \leqslant C^*_\varphi$ then for all copulas D with $C \leqslant D \leqslant C^*_\varphi$ we have $D^*_\varphi = C^*_\varphi$.

Observe that for each jointly strictly monotone copula C (i.e., $C(x,y) < C(x^*,y^*)$ whenever $x < x^*$ and $y < y^*$) the diagonal section $\delta_C : [0,1] \to [0,1]$ given by $\delta_C(x) = C(x,x)$ is an element of Φ. Moreover, if C is also associative then C is δ_C-invariant. In this statement, the associativity assumption may not be dropped: the copula C given by $C(x,y) = \frac{1}{2}(\min(x,y) + \max(x+y-1,0))$ is jointly strictly monotone but not δ_C-invariant.

Now we first are interested under which conditions C_φ is a copula and under which conditions a copula C is φ-invariant. Observe that, if $p \in]0,\infty[$ and $\varphi_p \in \Phi$ is defined by $\varphi_p(x) = x^p$, then the product Π is φ_p-invariant for each $p \in]0,\infty[$, whereas the Fréchet–Hoeffding lower bound W is φ_p-invariant only if $p = 1$, and W_{φ_p} is a copula only if $p \in]0,1]$.

As a consequence of [9, Theorem 7] we have: if C is an associative copula and $\varphi \in \Phi$, then C_φ is a copula if and only if for all $x, y, z \in [0,1]$ we have $|\varphi^{-1}(C(x,z)) - \varphi^{-1}(C(y,z))| \leqslant |\varphi^{-1}(x) - \varphi^{-1}(y)|$.

The concave elements in Φ have the remarkable property that they transform each copula into a copula:

Theorem 2.1. *For each $\varphi \in \Phi$ the following are equivalent*:

(i) *The function φ is concave.*
(ii) *For each copula C the function C_φ is a copula.*

3. Archimax copulas

Note that, because of the Lipschitz continuity, a copula C is φ_p-invariant for each $p \in]0, \infty[$ if and only if C is $\varphi_{1/n}$-invariant for each $n \in \mathbb{N}$. Following [6] (compare also [4]), a copula C^* is said to be the *maximum attractor of the copula* C (or, equivalently, C *belongs to the maximum domain of attraction of* C^*) if for all $(x, y) \in [0, 1]^2$ we have $\lim_{n\to\infty} C^n(x^{1/n}, y^{1/n}) = C^*(x, y)$.

Evidently, each copula C which is φ_p-invariant for each $p \in]0, \infty[$ is a maximum attractor of itself, i.e., $C^* = C$. The set of all maximum attractor copulas will be denoted by $\mathcal{M}$. Putting

$$\mathcal{A} = \{A : [0, 1] \to [0, 1] \mid A \text{ is convex and } \max(x, 1 - x) \leqslant A(x) \text{ for all } x \in [0, 1]\},$$

from [11,14] (compare also [5]) we know that each maximum attractor copula C^* can be expressed in the form

$$C^*(x, y) = \mathrm{e}^{\log(xy)\cdot A((\log x)/\log(xy))} \tag{3}$$

for some $A \in \mathcal{A}$. Evidently, Π is the weakest maximum attractor and M is the strongest one. The class $\mathcal{M}$ is closed under suprema and weighted geometric means. Although W belongs to the maximum domain of attraction of Π, there are copulas not belonging to any maximum domain of attraction.

Example 1 [3]. For the strict copula C whose additive generator $t : [0, 1] \to [0, \infty]$ is given by

$$t(x) = \log^2 x + 2^{n-5} \sin \frac{\log^2 x}{2^n} \quad \text{if } n \in \mathbb{Z} \text{ and } 2^{n+1}\pi \leqslant \log^2 x < 2^{n+2}\pi,$$

$\lim_{n\to\infty} C^n(x^{1/n}, y^{1/n})$ does not exist for, e.g., $(x, y) = (\frac{1}{2}, \frac{1}{2})$.

Now we clarify the relationship between φ_p-invariant copulas and the class $\mathcal{M}$ of maximum attractors (compare [1]):

Proposition 3.1. *For a copula C, the following are equivalent*:

(i) $C \in \mathcal{M}$.
(ii) $C_{\varphi_p} = C$ *for all* $p \in]0, \infty[$.
(iii) $C_{\varphi_p} = C_{\varphi_q} = C$ *for some* $p, q \in]0, \infty[$ *such that* $\frac{\log p}{\log q}$ *is irrational.*

Observe that $C_{\varphi_p} = C$ for some single $p \in]0, \infty[$ is not sufficient to guarantee $C \in \mathcal{M}$ (see Example 1 for $p = 2$).

If $t : [0, 1] \to [0, \infty]$ is a convex, decreasing bijection (and, therefore, an additive generator of some strict copula $C_{(t)}$) and if $A \in \mathcal{A}$ then the copula $C_{t,A}$ defined by

$$C_{t,A}(x, y) = t^{-1}\left(\big(t(x) + t(y)\big) \cdot A\left(\frac{t(x)}{t(x) + t(y)}\right)\right) \tag{4}$$

was called an *Archimax copula* in [4]. It is obvious that the class $\mathcal{A}_t = \{C_{t,A} \mid A \in \mathcal{A}\}$ contains both M and the strict copula $C_{(t)}$, and we always have $C_{(t)} \leqslant C_{t,A} \leqslant M$. Moreover, $\mathcal{M} = \mathcal{A}_{-\log}$ (note that $C_{(-\log)} = \Pi$).

For a fixed convex, decreasing bijection $t : [0, 1] \to [0, \infty]$ and for $p \in]0, \infty[$ define $\tau_p : [0, 1] \to [0, 1]$ by $\tau_p(x) = t^{-1}(p \cdot t(x))$. Evidently, $\tau_p \in \Phi$ for each $p \in]0, \infty[$. Note that a strict copula is $C_{(t)}$ is φ-invariant with respect to some $\varphi \in \Phi$ if and only if $t \circ \varphi = p \cdot t$, i.e., if $\varphi = \tau_p$ for some $p \in]0, \infty[$.

In complete analogy to Proposition 3.1 we have:

Corollary 3.2. *Let $t : [0, 1] \to [0, \infty]$ be a convex, decreasing bijection such that t^p is not convex whenever $p \in]0, 1[$. Then for each copula C the following are equivalent*:

(i) $C_{\tau_p} = C$ *for each* $p \in]0, \infty[$.

(ii) $C_{\tau_p} = C_{\tau_q} = C$ *for some* $p, q \in]0, \infty[$ *such that* $\frac{\log p}{\log q}$ *is irrational.*

Proposition 3.3. *Each Archimax copula* $C_{t,A}$ *is* τ_p*-invariant for each* $p \in]0, \infty[$.

However, not each copula which is τ_p-invariant for each $p \in]0, \infty[$ is an element of $\mathcal{A}_t$: take the function $t : [0, 1] \to [0, \infty]$ given by $t(x) = \log^2 x$ (which generates some *Gumbel copula*, see [10, Table 4.1, (4.2.4)] and [7]); then $\tau_p(x) = x^{\sqrt{p}}$ and Π is τ_p-invariant for each $p \in]0, \infty[$, but $\Pi \notin \mathcal{A}_t$.

Putting $\mathcal{B} = \{\mathcal{A}_t \mid t : [0, 1] \to [0, \infty]$ is a convex, decreasing bijection$\}$, we can determine maximal elements of $\mathcal{B}$:

Theorem 3.4. *Let* $t : [0, 1] \to [0, \infty]$ *be a convex, decreasing bijection and define* $\lambda^* = \inf\{\lambda \in]0, 1] \mid t^\lambda$ *is convex*$\}$. *Then* $\mathcal{A}_{t^{\lambda^*}}$ *is a maximal element of* $\mathcal{B}$ *with the property that all of its elements are* τ_p*-invariant for each* $p \in]0, \infty[$.

Example 2. *Consider the convex, decreasing bijection* $t : [0, 1] \to [0, \infty]$ *defined by* $t(x) = \frac{1}{x} - 1$ *(observe that it satisfies* $t'(1^-) = -1$*) which generates the copula* $C_{(t)}$ *given by* $C_{(t)} = \frac{xv}{x+y-xy}$. *The corresponding family* $(\tau_p)_{p \in]0,\infty[}$ *is then determined by* $\tau_p(x) = \frac{x}{p+(1-p)x}$. *Therefore,* $C_{(t)}$ *is the weakest copula which is* τ_p*-invariant for all* $p \in]0, \infty[$*, and each* $C_{t,A} \in \mathcal{A}_t$ *is given by*

$$C_{t,A}(x, y) = \frac{xy}{xy + A((1-x)y/(x+y-2xy)) \cdot (x+y-2xy)}.$$

Note that the functions τ_p *are multiplicative generators of the family of Ali–Mikhail–Haq copulas* [2,10].

Evidently, $C_{(t^{\lambda^*})}$ is the weakest associative copula which is τ_p-invariant for all $p \in]0, \infty[$. Whether it is also the weakest copula which is τ_p-invariant for all $p \in]0, \infty[$ is still an open problem. As a partial answer to this we have the following result:

Theorem 3.5. *Let* $t : [0, 1] \to [0, \infty]$ *be a convex, decreasing bijection such that* $t'(1^-) \neq 0$. *Then* $C_{(t)}$ *is the weakest copula which is* τ_p*-invariant for all* $p \in]0, \infty[$.

References

[1] J. Aczél, C. Alsina, Characterizations of some classes of quasilinear functions with applications to triangular norms and to synthesizing judgements, Methods Oper. Res. 48 (1984) 3–22.

[2] M.M. Ali, N.N. Mikhail, M.S. Haq, A class of bivariate distributions including the bivariate logistic, J. Multivariate Anal. 8 (1978) 405–412.

[3] C. Alsina, M.J. Frank, B. Schweizer, Associative functions on intervals: a primer on triangular norms, in press.

[4] P. Capéraà, A.-L. Fougères, C. Genest, Bivariate distributions with given extreme value attractor, J. Multivariate Anal. 72 (2000) 30–49.

[5] I. Cuculescu, R. Theodorescu, Extreme value attractors for star unimodal copulas, C. R. Math. Acad. Sci. Paris 334 (2002) 689–692.

[6] J. Galambos, The Asymptotic Theory of Extreme Order Statistics, Krieger, Melbourne, FL, 1987.

[7] C. Genest, L.-P. Rivest, A characterization of Gumbel's family of extreme value distributions, Statist. Probab. Lett. 8 (1989) 207–211.

[8] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic, Dordrecht, 2000.

[9] P. Mikusiński, M.D. Taylor, A remark on associative copulas. Comment, Math. Univ. Carolin. 40 (1999) 789–793.

[10] R.B. Nelsen, An Introduction to Copulas, Lecture Notes in Statist., vol. 139, Springer, New York, 1999.

[11] J. Pickands, Multivariate extreme value distributions, Bull. Inst. Internat. Statist. 49 (1981) 859–878 (with a discussion, 894–902).

[12] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, North-Holland, New York, 1983.

[13] A. Sklar, Fonctions de répartition à n dimensions et leurs marges, Publ. Inst. Statist. Univ. Paris 8 (1959) 229–231.

[14] J.A. Tawn, Bivariate extreme value theory: models and estimation, Biometrika 75 (1988) 397–415.