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The spectral geometry of the equatorial Podleś sphere

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Abstract

We propose a slight modification of the properties of a spectral geometry à la Connes, which allows for some of the algebraic relations to be satisfied only modulo compact operators. On the equatorial Podleś sphere we construct $\mathcal{U}_q(\mathrm{su}(2))$-equivariant Dirac operator and real structure which satisfy these modified properties. ***To cite this article: L. Dąbrowski et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

La géométrie spectrale de la sphère « équatoriale » de Podleś. Nous présentons une version légèrement modifiée des axiomes de la géométrie spectrale (réelle) au sens de Connes, qui permettent aux relations algébriques d'être satisfaites modulo les opérateurs compacts. Nous montrons que la sphère quantique « équatoriale » de Podleś est une géométrie spectrale et nous déterminons l'opérateur de Dirac et la structure réelle correspondante. ***Pour citer cet article : L. Dąbrowski et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Introduction

Recent examples of noncommutative spectral geometries on spaces coming from quantum groups [2,7,4,14,11,8] have opened a new interesting and promising area of research. Along these lines, by introducing a slight modification of the defining properties of a noncommutative geometry, we present a construction of a spectral geometry for the equatorial Podleś sphere S_q^2 of [13]. Both the Dirac operator D and the real structure J will be equivariant under the action of $\mathcal{U}_q(\mathrm{su}(2))$ on S_q^2.

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With q a real number, $0 < q \leqslant 1$, we denote by $\mathcal{A}(S_q^2)$ the algebra of polynomial functions generated by operators a, a^* and $b = b^*$, which satisfy the following commutation rules:

$$ba = q^2 ab, \quad a^* b = q^2 ba^*, \quad a^* a + b^2 = 1, \quad q^2 aa^* + q^{-2} b^2 = q^2.$$

This algebra contains a S^1-worth of classical points (the 'equator') given by the one dimensional representations $b = 0, a = \lambda$ with $\lambda \in S^1$.

The symmetry of the sphere, which we shall use for the equivariance, is the Hopf algebra module structure with respect to the $\mathcal{U}_q(\mathrm{su}(2))$ Hopf algebra and derived from the canonical $\mathcal{U}_q(\mathrm{su}(2))$ action on the $\mathcal{A}(\mathrm{SU}_q(2))$ algebra. Explicitly, the generators e, f, k of $\mathcal{U}_q(\mathrm{su}(2))$ act on the generators of $\mathcal{A}(S_q^2)$ in the following way:

$$\begin{aligned} &k \triangleright a = qa, \quad k \triangleright b = b, \quad e \triangleright a = -(1+q^2) q^{-5/2} b, \quad e \triangleright b = q^{1/2} a^*, \\ &f \triangleright a = 0, \quad f \triangleright b = -q^{3/2} a, \quad k \triangleright a^* = q^{-1} a^*, \quad e \triangleright a^* = 0, \quad f \triangleright a^* = (1+q^2) q^{-3/2} b. \end{aligned} \tag{1}$$

We shall use the fact that the irreducible finite dimensional representations of the Hopf algebra $\mathcal{U}_q(\mathrm{su}(2))$ are labelled by a positive half-integers (see [10], for example) and each representation space V_l has a basis $\{|l, m\rangle,\ m \in \{-l, -l+1, \ldots, l\}\}$ declared to be orthonormal.

2. Variations on spectral geometry

A spectral geometry (à la Connes) are data $(\mathcal{A}, \pi, \mathcal{H}, \gamma, J, D)$ fulfilling a series of requirements [3].

On the equatorial sphere $\mathcal{A} = \mathcal{A}(S_q^2)$ we construct an equivariant spectral geometry [16], starting from an equivariant representation π on a suitable Hilbert space $\mathcal{H}$. On the latter there are an equivariant real structure J and an equivariant Dirac operator D. However, with such a J it is not possible to satisfy all the requirements of [3]. Nevertheless, as we shall see, the algebraic requirements shall be obeyed up to compact operators. In particular, the antilinear isometry J which provides the real structure, will map $\pi(\mathcal{A})$ to its commutant only modulo compact operators.

$$\forall x, y \in \mathcal{A}: \quad \left[\pi(x), J\pi(y)J^{-1}\right] \in \mathcal{K}, \tag{2}$$

and the first order condition is valid only modulo compact operators,

$$\forall x, y \in \mathcal{A}: \quad \left[J\pi(x)J^{-1}, \left[D, \pi(y)\right]\right] \in \mathcal{K}. \tag{3}$$

The essentially unique Dirac operator, which comes out as the solution of the above condition, shall have the crucial property that all commutators $[D, \pi(x)]$, $x \in \mathcal{A}(S_q^2)$, are bounded.

3. The equivariant geometry of $\mathcal{A}(S_q^2)$

A fully equivariant approach, both for the real structure and the Dirac operator was worked out for the standard Podleś sphere in [7]. Here we build up on this and an earlier approach [9] (see also [12]) to construct an equivariant spectral geometry of the Equatorial Podleś sphere.

The starting ingredient of the equivariant spectral geometry, which we are about to construct is a equivariant representation of $\mathcal{A}(S_q^2)$ on a Hilbert space $\mathcal{H}$. Let us recall that an H-equivariant representation of an H-module algebra $\mathcal{A}$ on $\mathcal{H}$, is a representation π of $\mathcal{A}$ on $\mathcal{H}$ and a representation ρ of H on a dense subspace of $\mathcal{H}$ such that for every $a \in \mathcal{A}$ and $h \in H$ and v from a dense subspace we have:

$$\rho(h)\pi(a)v = \pi(h_{(1)} \triangleright a)\rho(h_{(2)})v. \tag{4}$$

In our case, this is the same as a representation of the crossed product of $\mathcal{U}_q(\mathrm{su}(2)) \ltimes \mathcal{A}(S_q^2)$ on a dense subspace of the Hilbert space. The general theory for the entire family of Podleś spheres is in [15].

Proposition 3.1. *There exists two irreducible $\mathcal{U}_q(\mathrm{su}(2))$-equivariant representations (denoted by $\pi_\pm$) of the algebra $\mathcal{A}(S_q^2)$ on the Hilbert space $\mathcal{H}_h = \bigoplus_{l=\frac{1}{2},\frac{3}{2},\ldots} V_l$ given by:*

$$\begin{aligned}
\pi_\pm(a)|l,m\rangle &= \pm(1+q^2)\frac{q^{m-1/2}}{[2l][2l+2]}\sqrt{[l+m+1][l-m]}\,|l,m+1\rangle \\
&\quad + \frac{q^{m-l-1/2}}{[2l+2]}\sqrt{[l+m+1][l+m+2]}\,|l+1,m+1\rangle \\
&\quad - \frac{q^{m+l+1/2}}{[2l]}\sqrt{[l-m][l-m-1]}\,|l-1,m+1\rangle, \\
\pi_\pm(b)|l,m\rangle &= \pm\frac{1}{[2l][2l+2]}\big([l-m+1][l+m]-q^2[l-m][l+m+1]\big)|l,m\rangle \\
&\quad - \frac{q^{m+1}}{[2l+2]}\sqrt{[l-m+1][l+m+1]}\,|l+1,m\rangle - \frac{q^{m+1}}{[2l]}\sqrt{[l-m][l+m]}\,|l-1,m\rangle,
\end{aligned} \tag{5}$$

with $\pi(a^*)$ being the Hermitian conjugate of $\pi(a)$ and $[x] := (q-q^{-1})^{-1}(q^x-q^{-x})$.

The proof is a long but straightforward calculation based on the covariance property (4) with the natural representation ρ of $\mathcal{U}_q(\mathrm{su}(2))$ on $\mathcal{H}_h$ and the $\mathcal{U}_q(\mathrm{su}(2))$-module structure of $\mathcal{A}(S_q^2)$ given in (1). The representations $\pi_\pm$ are equivalent to the left regular representation of $\mathcal{A}(\mathrm{SU}_q(2))$ on $L^2(\mathrm{SU}_q(2))$ (with the Haar measure) when this representation is restricted to the subalgebra $\mathcal{A}(S_q^2)$ and the representation space is restricted to the (L^2-completion) of certain vector spaces (left $\mathcal{A}(S_q^2)$-modules) constructed in [1].

We take as the Hilbert space of our geometry $\mathcal{H} = \mathcal{H}_h \oplus \mathcal{H}_h$, with the natural grading $\gamma = \mathrm{id} \oplus (-\mathrm{id})$ and the representation $\pi(x) = \pi_+(x) \oplus \pi_-(x)$ for any $a \in \mathcal{A}(S_q^2)$, which is equivariant with respect to the diagonal action (which we call again ρ) of $\mathcal{U}_q(\mathrm{su}(2))$ on $\mathcal{H}$. As in [7], we follow the method of equivariance to find first the real structure J. Let us recall, that J is the antiunitary part of an antilinear operator T on $\mathcal{H}$, which must then satisfy for any $h \in \mathcal{U}_q(\mathrm{su}(2))$ (on a dense subspace of $\mathcal{H}$), $\rho(h)T = T\rho(Sh)^*$. Taking into account the required commutation relations with the grading γ, that is $\gamma J = -J\gamma$ and $J^2 = -1$, one easily obtain that J must be,

$$J|l,m\rangle_\pm = \mathrm{i}^{2m}|l,-m\rangle_\mp, \tag{6}$$

where the label $\pm$ refers to the two copies of $\mathcal{H}_h$ which are marked by the eigenvalues of γ. With this data we immediately meet an obstruction:

Proposition 3.2. *The operator J defined above does not satisfy the 'commutant' requirement of a real spectral triple, that is, there exist $x \in \mathcal{A}(S_q^2)$ for which $J\pi(x)J^{-1}$ is not in the commutant of $\pi(\mathcal{A}(S_q^2))$.*

Then, we move on to look for a variation of spectral geometry up to infinitesimal, as introduced earlier. Let $\mathcal{K}$ denote the ideal of compact operators on $\mathcal{H}$ and $\mathcal{K}_q \subset \mathcal{K}$ be the ideal generated by operators L_q of the form $L_q|l,m\rangle_\pm = q^l|l,m\rangle_\pm$.

Proposition 3.3. *The operator J, defined in (6), maps $\pi(\mathcal{A}(S_q^2))$ to its commutant modulo compact operators (in fact modulo $\mathcal{K}_q$). More precisely, for any $x, y \in \mathcal{A}(S_q^2)$,*

$$\big[\pi(x), J\pi(y)J^{-1}\big] \in \mathcal{K}_q. \tag{7}$$

To prove this proposition, it is convenient to use compact perturbations of the representation π. Exact formulæ shall be contained in the extended version of this Note.

As a next step we derive the Dirac operator D. Beside postulating that D anticommutes with γ and commutes with J, we shall also require that D is equivariant, that is it commutes with the representation ρ of $\mathcal{U}_q(\mathrm{su}(2))$ on $\mathcal{H}$. Each operator satisfying these condition must be of the form

$$D|l,m\rangle_{\pm} = d_l^{\pm}\,|l,m\rangle_{\mp}, \quad d_l^{\pm} \in \mathbb{R}. \tag{8}$$

Proposition 3.4. *Up to rescaling and addition of an odd constant as well as of an odd element from $\mathcal{K}_q$, there exists only one operator D of the form* (8) *which satisfies the order-one condition up to compact operators* (*in fact modulo $\mathcal{K}_q$*)*, that is for all $x, y \in \mathcal{A}(S_q^2)$,*

$$\big[J\pi(x)J^{-1}, \big[D, \pi(y)\big]\big] \in \mathcal{K}_q. \tag{9}$$

For this operator D, the parameters $d_l^{\pm}$ are given by $d_l^+ = d_l^- = l + \frac{1}{2}$.

The condition (9) has been verify explicitly for all pairs of generators of $\mathcal{A}(S_q^2)$ with the help of a symbolic computation program. Furthermore, we have,

Proposition 3.5. *For any $x \in \mathcal{A}(S_q^2)$, the commutators $[D, \pi(x)]$ are bounded.*

It is evident that the operator D is self-adjoint on a natural domain in $\mathcal{H}$ and that its resolvent is compact. Since the spectrum of $|D|$ consists only of eigenvalues $k = l + \frac{1}{2} \in \mathbb{N}$ with multiplicity $4k$, we managed to realize the suggestion in [5] to use D with a classical spectrum. Thus, the deformation being isospectral, the dimension requirement is satisfied with the spectral dimension of $(\mathcal{A}(S_q^2), \mathcal{H}, D)$ being $n = 2$.

We have made some advancement in the study of the q-geometry and expect that similar structures exist on other q-deformed spaces. There are still many points to be addressed, notable the existence of a volume form and the fulfillment of other axioms of spectral geometries. These points shall be addressed in the extended version of the note. A spectral triple for $\mathrm{SU}_q(2)$ which is isospectral is presented in [6].

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