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Harmonic Analysis

Bellman function for the estimates of Littlewood–Paley type and asymptotic estimates in the $p-1$ problem

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Abstract

We utilize the method of Bellman functions to derive new L^p-estimates of Littlewood–Paley type involving $p-1$. Among the applications to singular integrals we improve the $2(p-1)$ bounds for the Ahlfors–Beurling operator on $L^p(\mathbb{C})$ when $p \to \infty$. In addition, dimensionless estimates of Riesz transforms in the classical as well as in the Ornstein–Uhlenbeck setting are attained. ***To cite this article: O. Dragičević, A. Volberg, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Fonction de Bellman pour obtenir des estimations de type Littlewood–Paley et de type assymptotique pour le problème $p-1$. On utilise la technique de la fonction de Bellman pour obtenir les estimations nouvelles et assez générales du type de Littlewood–Paley. Comme la premier consequence de nos estimation du type de Littlewood–Paley on derive les resultats classiques concernants les bornes libre de dimension pour les transformations de Riesz. La deuxième consequence est une amilioration de la borne dans $L^p(\mathbb{C})$ de transformation de Ahlfors–Beurling quand $p \to \infty$. ***Pour citer cet article : O. Dragičević, A. Volberg, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Introduction and main objects

The main purpose of this Note is to summarize certain estimates of Littlewood–Paley type on L^p in which $p-1$ appears as a key factor. They were derived in cases of different semigroup extensions. The method exploited, i.e.

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the Bellman function, often tends to give precise estimates. These inequalities are then applied to different concrete singular integrals, where the results obtained are in some cases best known to date.

Let us begin by setting up some notation.

Poisson extensions: take a function $f \in C_c(\mathbb{R}^n)$ and denote by Pf the continuous prolongation of f onto $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$ which coincides with f on the boundary and satisfies the Laplace equation $\Delta\varphi = 0$ in the upper half-space (here Δ stands for the self-adjoint closure of the usual Laplacian). Equivalently, it is given by a semigroup generator $A = \sqrt{-\Delta}$, i.e. $Pf(x,t) = \mathrm{e}^{-tA} f(x)$.

For a given f on the plane, its *heat extension* to the upper half-space $\mathbb{R}^3_+$ can be defined by the formula $Hf(x,t) := \mathrm{e}^{t\Delta} f(x)$. Such a function solves the heat equation $\Delta\varphi = \partial_t\varphi$ in $\mathbb{R}^3_+$, whereas on the boundary it coincides with f.

Among the main driving forces behind our quest for $p-1$ estimates is the *Ahlfors–Beurling operator* T. It is defined as a principal value integral by

$$Tf(z) = -\frac{1}{\pi} \int\limits_{\mathbb{C}} \frac{f(\zeta)}{(z-\zeta)^2} \,\mathrm{d}A(\zeta).$$

There is a long-standing conjecture by Iwaniec [5] which asserts that for $p \in (1,\infty)$,

$$\|T\|_{B(L^p(\mathbb{C}))} = p^* - 1, \tag{1}$$

where $p^* = \max\{p, p'\}$. The roots of this problem lie deeply in the theory of quasiconformal mappings, where it has been known for decades that T plays an essential rôle.

Classical *Riesz transforms* R_k, $k = 1,\dots,n$, on $L^p(\mathbb{R}^n)$ can be introduced symbolically as $R_k = \partial_k \circ A^{-1}$. They are Fourier multipliers with symbols $\mathrm{i}x_k/\|x\|$. Their connection to the Ahlfors–Beurling operator is revealed by the well-known identity $T = R_2^2 - R_1^2 + 2\mathrm{i}R_1R_2$, where R_1 and R_2 are the Riesz transforms on the plane.

Ornstein–Uhlenbeck Riesz transforms: in case when $\mathbb{R}^n$ is endowed with the normalized Gaussian measure $\mathrm{d}\mu(x) = (2\pi)^{-n/2}\,\mathrm{e}^{-|x|^2/2}\,\mathrm{d}x$, a natural way to obtain a symmetric operator related to Δ is to consider the *Ornstein–Uhlenbeck Laplacian*

$$\Delta_{\mathrm{OU}} := \Delta - \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}.$$

The same symbol will be used to denote its closure, which is self-adjoint in the Gaussian L^2. This operator generates the third type of an extension that we are going to consider. Let the symbol Ψf stand for the 'harmonic' continuation of a function f from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}_+$ generated by Δ_{OU}. Explicitly, $\Psi f(x,t) = \mathrm{e}^{-t\mathbf{A}} f(x)$ where $\mathbf{A} = \sqrt{-\Delta_{OU}}$. This is clearly an analogue to the Poisson extension in the classical setting. The associated Riesz transforms are given formally by $\mathbf{R}_k = \partial_k \circ \mathbf{A}^{-1}\pi_0$, where π_0 is the orthogonal projection $L^2(\mathbb{R}^n,\mu) \to \{f;\ \int_{\mathbb{R}^n} f\,\mathrm{d}\mu = 0\}^\perp$.

For a differentiable, $\mathbb{R}^m$-valued function $\varphi = (\varphi_1,\dots,\varphi_m)$ defined on some domain $\Omega \subset \mathbb{R}^n$, we introduce its *Jacobi matrix* $J\varphi$ as usual: $J\varphi(\omega) = [\partial_j\varphi_i(\omega)]_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$ for each $\omega \in \Omega$. Finally, by $\|\cdot\|_{\mathrm{HS}}$ we will denote the Hilbert–Schmidt norm of a matrix.

2. Results

Our first results are the following Littlewood–Paley-type inequalities (LPI) which we derive in cases of all three semigroup extensions described in the previous section. In all statements to follow, p and q are going to be conjugate exponents from $(1,\infty)$.

Theorem 2.1 (LPI for Poisson kernel). *Let M, N, n be arbitrary natural numbers. Take test functions $f \in L^p(\mathbb{R}^n \to \mathbb{C}^M)$ and $g \in L^q(\mathbb{R}^n \to \mathbb{C}^N)$. Then*

$$2\int_0^\infty\int_{\mathbb{R}^n} \left\|J(Pf)(x,t)\right\|_{\mathrm{HS}}\left\|J(Pg)(x,t)\right\|_{\mathrm{HS}}\, t\,\mathrm{d}x\,\mathrm{d}t \leqslant (p^*-1)\|f\|_p\|g\|_q.$$

Theorem 2.2 (LPI for Gaussian Poisson kernel). *There is an absolute constant $C > 0$, such that for all test functions f, g and $1 < p < \infty$,*

$$\int_0^\infty\int_{\mathbb{R}^n} \left\|J(\Psi f)(x,t)\right\|_{\mathrm{HS}}\left\|J(\Psi g)(x,t)\right\|_{\mathrm{HS}} t\,\mathrm{d}\mu(x)\,\mathrm{d}t \leqslant C(p^*-1)\|f\|_{L^p_\mu}\|g\|_{L^q_\mu}.$$

Theorem 2.3 (LPI for heat kernel). *For any $f, g \in C_c^\infty(\mathbb{R}^2)$ we have*

$$2\int_0^\infty\int_{\mathbb{R}^2} \left\|J(Hf)(x,t)\right\|_{\mathrm{HS}}\left\|J(Hg)(x,t)\right\|_{\mathrm{HS}}\,\mathrm{d}x\,\mathrm{d}t \leqslant (p^*-1)\|f\|_p\|g\|_q.$$

The proofs of these theorems rely heavily on two things: (i) sharp properties of the function we bring up in Theorem 2.5 that are closely related to the result of Burkholder [2] involving differential subordination of martingales; (ii) an elementary lemma from linear algebra, which is slightly unexpected. We are deeply grateful to S. Szarek who supported our belief in the truth of this lemma at the moment of doubt. Let us start with it.

Lemma 2.4. *Suppose $\mathcal{H}$ is a finite-dimensional real Euclidean space; $\mathcal{H}_i$, $i = 1, 2$, are two non-trivial mutually orthogonal subspaces of $\mathcal{H}$ and P_i are the corresponding orthogonal projections. Let T be a self-adjoint operator such that $\langle Th, h\rangle \geqslant 2\|P_1h\|\|P_2h\|$ for all $h \in \mathcal{H}$. Then there exists $\tau > 0$, satisfying*

$$\langle Th, h\rangle \geqslant \tau\|P_1h\|^2 + \tau^{-1}\|P_2h\|^2$$

again for all $h \in \mathcal{H}$. Hence for any Hilbert–Schmidt operator L, acting from any space (possibly infinite-dimensional) into $\mathcal{H}$, we have

$$\mathrm{tr}(L^*TL) \geqslant 2\|P_1L\|_{\mathrm{HS}}\|P_2L\|_{\mathrm{HS}}.$$

In order to define properly the function stemming from (i) above we need some further notation. So let M, N be natural numbers and define $\Omega = \{(\zeta, \eta, Z, H) \in \mathbb{C}^M \times \mathbb{C}^N \times \mathbb{R} \times \mathbb{R};\ |\zeta|^p < Z, |\eta|^q < H\}$.

Theorem 2.5. *There is a function $B : \overline{\Omega} \to \mathbb{R}$, such that $0 \leqslant B(\zeta, \eta, Z, H) \leqslant (p^*-1)Z^{1/p}H^{1/q}$ everywhere on its domain, and for any $a_\pm = (\zeta_\pm, \eta_\pm, Z_\pm, H_\pm) \in \overline{\Omega}$,*

$$B\left(\frac{a_+ + a_-}{2}\right) - \frac{B(a_+) + B(a_-)}{2} \geqslant \left|\frac{\zeta_+ - \zeta_-}{2}\right| \cdot \left|\frac{\eta_+ - \eta_-}{2}\right|. \tag{2}$$

This is our *Bellman function*. We are justified in calling it this way, because it imitates the behavior of Bellman functions from the theory of Optimal Control, see e.g. [6]. Just that in our case it is *Stochastic* Optimal Control, which is reflected in the fact that the key property of B, namely (2), is a *second order* differential inequality which plays the role of the second order Bellman PDE for stochastic Bellman function (classical Bellman functions satisfy the first order Bellman PDE). Readers interested in stochastic Bellman function theory are referred to [6], while those interested in relations between Bellman functions in Harmonic Analysis and Bellman functions in Stochastic

Optimal Control should consult the survey paper [9]. The concept of the Bellman function in Harmonic Analysis first appeared in the preprint version of [8] in 1995 and was developed into a sharp tool in [3,4,7,8,10,11]. It is explained in [9] how one can 'guess' (2) from the form of the inequality one needs to prove. In our present case this is our LPI.

We conclude by displaying our $p-1$ estimates for certain singular integrals. They were derived as consequences of Theorems 2.1–2.3 and, in a sense, are their weaker versions. Still, some of the results we obtained are the best known at present. The first one concerns the Ahlfors–Beurling operator.

Theorem 2.6.

$$\limsup_{p^*\to\infty}\frac{\|T\|_p}{p^*-1}\leqslant\sqrt{2}\quad \textit{and}\qquad \lim_{p^*\to\infty}\frac{\|T\|_{L^p_{\mathrm{real}}}}{p^*-1}=1.$$

The inequality $\|T\|_p\leqslant 2(p^*-1)$ was first proven in [10] and shortly afterwards independently in [1]. The two other inequalities above are asymptotically better and represent additional evidence in favour of Iwaniec's conjecture (1).

The last application involves dimension-free estimates for Riesz transforms. The first such result was obtained by E. Stein. In the Gaussian case we actually reprove the classical result of P.A. Meyer obtained first by probabilistic methods. G. Pisier presented later an analytic proof.

Theorem 2.7. *If* $1<p<\infty$ *and* $f\in L^p(\mathbb{R}^n)$*, then*

$$\|R_1f,\dots,R_nf\|_{L^p(\mathbb{R}^n\to\mathbb{C}^n)}\leqslant 2(p^*-1)\|f\|_{L^p(\mathbb{R}^n)},$$

and therefore also $\|R_k\|_{B(L^p)}\leqslant 2(p^*-1)$ *for* $k=1,\dots,n$*. In the case of the Ornstein–Uhlenbeck Riesz transforms there are constants* $C_p>0$*, not depending on the dimension* n*, for which*

$$\|\mathbf{R}_1f,\dots,\mathbf{R}_nf\|_{L^p_\mu(\mathbb{R}^n\to\mathbb{C}^n)}\leqslant C_p\|f\|_{L^p_\mu(\mathbb{R}^n)}.$$

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