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C. R. Acad. Sci. Paris, Ser. I 340 (2005) 677–682
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Differential Geometry

On the rigidity of hyperbolic cone-manifolds

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Received 11 March 2005; accepted 19 March 2005

Available online 27 April 2005

Presented by Étienne Ghys

Abstract

Starting with a compact hyperbolic cone-manifold of dimension $n \geqslant 3$, we study the deformations of the metric with the aim of getting Einstein cone-manifolds. If the singular locus is a closed codimension 2 submanifold and all cone angles are smaller than 2π, we show that there is no non-trivial infinitesimal Einstein deformations preserving the cone angles. ***To cite this article: G. Montcouquiol, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Sur la rigidité des cônes-variétés hyperboliques. Partant d'une cône-variété hyperbolique de dimension $n \geqslant 3$, on étudie les déformations de la métrique dans le but d'obtenir des cônes-variétés Einstein. Dans le cas où le lieu singulier est une sous-variété fermée de codimension 2 et que tous les angles coniques sont plus petits que 2π, on montre qu'il n'existe pas de déformations Einstein infinitésimales non triviales préservant les angles coniques. ***Pour citer cet article : G. Montcouquiol, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

Dans leur article [5], Hodgson et Kerckhoff montrent qu'il est impossible de déformer une cône-variété hyperbolique de dimension 3 vérifiant certaines hypothèses sans modifier ses angles coniques. Le principe de la démonstration est de réussir à appliquer la méthode de Calabi–Weil aux cônes-variétés : on montre que la représentation d'holonomie n'admet pas de déformations non triviales du type voulu. On rencontre ce faisant toutes les difficultés inhérentes à l'étude des cônes-variétés.

Dans le cas des variétés fermées, Koiso [6] a donné un analogue de la méthode de Calabi–Weil, qui étudie directement les déformations de la métrique (cf. aussi [1], Section 12.H). Cette méthode est plus facilement géné-

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2005.03.019

ralisable et s'applique, en dimension supérieure, à une classe de variétés plus vaste, à savoir les variétés Einstein à courbure suffisamment négative. L'objet de cette Note est d'adapter la méthode de Koiso pour démontrer qu'en dimension supérieure ou égale à 3, et sous des hypothèses voisines de celles du théorème de Hodgson et Kerckhoff, on ne peut pas déformer une cône-variété hyperbolique en des cônes-variétés Einstein sans en modifier les angles coniques.

On se place dans le cadre suivant : M est une variété compacte sans bord, et Σ est une sous-variété fermée plongée de codimension 2 ayant p composantes connexes. Une structure de *cône-variété* sur M, de lieu singulier Σ et d'angles coniques $\alpha_1, \ldots, \alpha_p$, est la donnée d'une métrique riemanienne g (non complète) sur $M \setminus \Sigma$ et d'une métrique riemannienne g_i sur chaque composante connexe Σ_i de Σ, de telle sorte que quand on se rapproche d'un point de Σ_i, g ressemble asymptotiquement au produit de g_i avec la métrique d'un cône (de dimension deux) d'angle au sommet α_i.

Les déformations infinitésimales d'une telle structure peuvent toujours se mettre sous une forme standard au voisinage du lieu singulier (dans notre cadre, ces déformations standards forment une famille de dimension infinie). En particulier, une déformation ne modifiant pas les angles a la propriété d'être L^2 à dérivée covariante L^2.

Une des difficultés dans l'étude des cônes-variétés est de pouvoir faire les intégrations par parties. On cite dans la Section 3 deux résultats dans ce sens.

Soit M une cône-variété hyperbolique, dont tous les angles coniques sont strictement inférieurs à 2π. Partant d'une déformation infinitésimale Einstein h_0 préservant les angles (donc à dérivée covariante L^2), la démonstration de sa trivialité se fait en deux temps. On veut d'abord se débarrasser des déformations triviales, et on cherche pour ce faire à imposer la condition de jauge de Bianchi. On a alors besoin de résoudre l'équation de normalisation $\beta \circ \delta^*(\alpha) = \beta(h_0)$, ce qui peut se faire en imposant de bonnes conditions sur la 1-forme α quand tous les angles coniques sont strictement inférieurs à 2π (Section 4).

On applique ensuite une technique de Bochner à la déformation normalisée $h = h_0 - \delta^*\alpha$. En utilisant la formule de Weitzenböck idoine et un premier résultat d'intégration par parties, on obtient $\delta^\nabla \mathrm{d}^\nabla h + (n-2)h = 0$, et une dernière intégration par parties permet de conclure que $h_0 = \delta^*\alpha$ (Section 5), et donc que l'on a bien rigidité infinitésimale relativement aux angles coniques au sein des cônes-variétés Einstein.

1. Introduction

In their celebrated article [5], Hodgson and Kerckhoff showed that it is not possible to deform a hyperbolic cone-manifold of dimension 3 (with some restrictions on its geometry) while keeping its cone angles fixed. Their results lead to many applications in the geometry of hyperbolic 3-manifolds, such as the geometrization of small orbifolds or the study of Kleinian groups [2,3].

The main idea of this theorem's proof is to apply the Calabi–Weil arguments to the setting of cone-manifolds: one shows that the holonomy representation has no non-trivial deformation of the required type. This is not done without some difficulties, which are unavoidable when one deals with cone-manifolds; we shall encounter them later.

In the closed manifold case, there exists a method due to Koiso [6] (cf. also [1] Section 12.H), similar to the Calabi–Weil arguments, which uses no longer the holonomy representation, but instead takes directly into account the deformation of the metric. This second method has the merit of being easier to generalize; it deals in larger dimensions with a wider class of manifolds, namely the Einstein ones (under some curvature restrictions).

The goal of this Note is then to adapt the Koiso arguments to prove the following theorem:

Theorem 1.1. *Let M be a compact hyperbolic cone-manifold of dimension $n \geqslant 3$, whose singular locus is a closed codimension 2 submanifold and whose cone angles are all strictly smaller than 2π. Then M is infinitesimally rigid among Einstein cone-manifolds with fixed cone angles.*

2. Cone-manifolds and their deformations

Cone-manifolds can be defined in different ways, depending on what kind of cone-manifolds and properties are being used. Constant curvature cone-manifolds are the easiest to describe, either geometrically, as a gluing of geodesic simplices, or by stating the metric explicitly. It is this last approach that we will use, despite the fact that we will deal mainly with hyperbolic cone-manifolds.

In the general case, the singular locus of a cone-manifold can be rather intricate, but for the purpose of our study we will restrict in this Note to the following setting.

Let M be a *closed* manifold of dimension $n \geqslant 3$, and let $\Sigma = \coprod_{i=1}^{p} \Sigma_i$ be a closed, embedded submanifold of codimension 2 (the Σ_i are the connected components of Σ). Throughout this text we will often use M to denote either M or, improperly, $M \setminus \Sigma$.

Definition 2.1. Let $\alpha_1, \ldots, \alpha_p$ be positive real numbers. The manifold M carries a *cone-manifold* structure, of singular locus $\Sigma = \coprod_{i=1}^{p} \Sigma_i$ and cone angles $\alpha_1, \ldots, \alpha_p$, if:

- $M \setminus \Sigma$ carries a Riemannian metric g, which is not complete,
- for all i between 1 and p, Σ_i carries a Riemannian metric g_i,
- for all i between 1 and p, every point x of Σ_i admits a neighborhood V in M diffeomorphic to $D^2 \times U$, where $U = V \cap \Sigma_i$ is a neighborhood of x in Σ_i. Using (local) cylindrical coordinates in this neighborhood, the metric g can be written as
$$g = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + g_i + q,$$
where the angle coordinate θ is not defined modulo 2π but modulo the cone angle α_i, and where q is a symmetric 2-tensor such that $g(q,q) = \mathrm{O}(r^2)$ and $g(\nabla q, \nabla q) = \mathrm{O}(r)$.

We can then define a hyperbolic cone-manifold as a cone-manifold such that the Riemannian metrics g and g_i are hyperbolic. In this case the tensor q defined above is $q = (\sinh(r)^2 - r^2)\,\mathrm{d}\theta^2 + (\cosh(r)^2 - 1)g_i$.

Let M be a cone-manifold like above, with metric g and singular locus Σ. Now let g_t be a smooth family of singular metrics, such that $g_0 = g$ and that for all t, g_t defines a cone-manifold structure on M with singular locus Σ.

Owing to the local expression of a cone metric, one can show that, up to modifying the family g_t by diffeomorphisms, the symmetric 2-tensor $h = \frac{\mathrm{d}g_t}{\mathrm{d}t}|_{t=0}$ is, in a neighborhood of the singular locus, a linear combination of four kinds of deformations, which modifies respectively the cone angles, the metric of the singular locus, the remainder q, and last, the way of 'gluing' the angle coordinate when passing from a coordinate chart to another.

What should be noted is that these four kinds of tensors ('standard infinitesimal deformations') are in L^2, but that only the last three have square-integrable covariant derivative. Hence, it is a L^2 property of the covariant derivative of a deformation that tells us whether it preserves the cone angles or not.

3. Two results of integration by parts

Throughout the rest of this text, M will denote a cone-manifold as defined above.

Since the smooth part of a cone-manifold is non-compact, it is impossible to use directly the Stokes theorem. Thus we have to prove the integration by parts results that will be needed later.

The first result is due to Cheeger [4]. Actually he did not state the following theorem in this form; he rather proved two results which combined (and slightly adapted) give a proof of the following, cf also the appendix of [5].

Theorem 3.1. *Let $\alpha \in \Omega^p M$ and $\beta \in \Omega^{p+1} M$ be two smooth forms on M such that α, $\mathrm{d}\alpha$, β, and $\delta\beta$ are in L^2. Then $\langle \alpha, \delta\beta \rangle = \langle \mathrm{d}\alpha, \beta \rangle$.*

The next result deals with tensors instead of forms, but the proof, although not similar, is basically an adaptation of the ideas of Cheeger mentioned above.

Theorem 3.2. *Let $u \in C^\infty(T^{(r,s)}M)$, $v \in C^\infty(T^{(r+1,s)}M)$ be two tensors such that u, ∇u, v, and $\nabla^* v$ are in L^2. Then $\langle u, \nabla^* v \rangle = \langle \nabla u, v \rangle$.*

4. Einstein deformations and the normalization equation

An *Einstein metric* is a Riemannian metric g which satisfies the equation $\mathrm{ric}(g) = cg$, where the left term is the Ricci curvature tensor and c is a constant. By rescaling the metric one can multiply c by any positive number, so what is only relevant here is the sign of c. All constant curvature metrics are Einstein, and in fact in dimension 3 these are the only ones. Thus the Einstein condition can be seen as a generalization, or a weakened form, of the constant curvature condition.

Since we are dealing with negatively curved cone-manifolds, and chiefly with hyperbolic ones, we will only consider Einstein metrics such that $E(g) = 0$, where $E(g) = \mathrm{ric}(g) + (n-1)g$, and the constant $n-1$ is choosen so that hyperbolic metrics satisfy this equation.

If g_t is a smooth family of Einstein metrics (i.e. satisfying $E(g_t) = 0$) on a given manifold M with $g_0 = g$, then the symmetric 2-tensor $h = \frac{\mathrm{d}}{\mathrm{d}t} g_t|_{t=0}$ satisfies the linearized Einstein equation $E'_g(h) = 0$. The computation of E'_g is classical, cf for instance [1], chapter 1 (and also for a detailed explanation of the involved operators):

$$E'_g(h) = \nabla_g^* \nabla_g h - 2\mathring{R}_g h - \delta_g^*(2\delta_g h + \mathrm{d}\,\mathrm{tr}_g\, h).$$

We will often omit the subscript g. Any symmetric 2-tensor h on M satisfying the equation $E'_g(h) = 0$ will be called an *infinitesimal Einstein deformation* of the (Einstein) manifold (M, g).

Now, if g is Einstein and ϕ is a diffeomorphism of M, then the metric $\phi^* g$ is also Einstein. Therefore, if ϕ_t is a smooth family of diffeomorphisms with $\phi_0 = Id$, the induced infinitesimal deformation $\frac{\mathrm{d}}{\mathrm{d}t} \phi_t^* g|_{t=0}$ will of course be Einstein. Such a deformation will be called *trivial*. The set of trivial deformations is easily shown to be equal to the image of the operator $\delta^* : \Omega^1 M \to \mathcal{S}^2 M$, which maps a form α to the symmetric 2-tensor defined by $\delta^*\alpha(x, y) = \frac{1}{2}((\nabla_x \alpha)(y) + (\nabla_y \alpha)(x))$, cf. [1], Section 1.60.

The usual way to get rid of trivial deformations is to impose a gauge condition on the infinitesimal deformations, which means to consider only deformations satisfying some equation. Here we will use the Bianchi gauge, that is, we want our deformations h to satisfy $\beta_g(h) = 0$, where the Bianchi operator (related to the metric g) $\beta_g : \mathcal{S}^2 M \to \Omega^1 M$ is defined by $\beta_g(h) = \delta_g h + \frac{1}{2}\mathrm{d}\,\mathrm{tr}_g\, h$. The operator δ appearing here is the adjoint of the operator δ^* introduced previously; alternatively, it is the restriction to $\mathcal{S}^2 M$ of ∇^*.

Thus, starting with an infinitesimal deformation h_0, we want to be able to modify it by a trivial deformation, in an essentially unique way, so that the result will satisfy the gauge condition. More precisely, we want to find a form α such that $\beta(h_0 - \delta^*\alpha) = 0$, or equivalently to solve the *normalization equation*

$$\beta \circ \delta^* \alpha = \beta(h_0).$$

Using the fact that $\nabla\alpha = \delta^*\alpha + \frac{1}{2}\mathrm{d}\alpha$ and the well-known Weitzenböck formula

$$\Delta\alpha = (\mathrm{d}\delta + \delta\mathrm{d})\alpha = \nabla^*\nabla\alpha + \mathrm{ric}(\alpha) = \nabla^*\nabla\alpha - (n-1)\alpha,$$

an easy computation gives $\beta \circ \delta^*\alpha = \frac{1}{2}(\nabla^*\nabla\alpha + (n-1)\alpha)$.

This operator is well-behaved, in particular it is elliptic, so it has good regularity properties: its solutions will be smooth as soon as the second term is. But sadly, since a cone-manifold is singular, we will not be able to use all the result of the standard theory of elliptic operators. Instead, we now have to investigate directly the properties of the solutions of the normalization equation. The goal is to solve this equation $L\alpha = \phi$ (with L a shorthand for $\frac{1}{2}(\nabla^*\nabla\alpha + (n-1)\alpha)$), while keeping enough information on the solution in order to be able to use a Bochner technique. This motivates the following theorem.

Theorem 4.1. *Let M be a hyperbolic cone-manifold whose cone angles are all strictly smaller than 2π. Let $\phi \in \Omega^1 M$ be a smooth 1-form which is also in $L^2(T^*M)$. Then there exists a unique form $\alpha \in \Omega^1 M$, solution to the equation $L\alpha = \phi$, such that α, $\nabla\alpha$, $\mathrm{d}\delta\alpha$, and $\nabla\,\mathrm{d}\alpha$ are in L^2.*

Using the framework of unbounded linear operators in Hilbert spaces, one can find several domains on which L is self-adjoint and thus invertible. Now we have to show that there is a domain with the required properties.

To prove this, one has to actually solve the equation $L\alpha = 0$ near the singular locus. This involves some book-keeping; the subtlest part is to find a suitable decomposition of α. Since the angle coordinate θ is only local, it is not possible to carry out a decomposition in Fourier series. However one can exhibit a Hilbert basis of eigenvectors of the Laplacian on functions on the boundary of a tubular neighborhood of the singular locus, such that any element of this basis is also an eigenvector for the derivative in the θ direction. Now, using some commutation relations (closely related to the Weitzenböck formula for 1-forms mentioned above), one can deduce from this a similar Hilbert basis for 1-forms.

So one can write α as a sum of functions of the radial coordinate r times these eigenvectors of the Laplacian. Thus we reduce the PDE to a family of linear differential equations, and we have a good knowledge of the behaviour of the solutions of the homogeneous equation. This behaviour is explicitly related to the values of the cone angles: the pointwise norm of a given solution near a component of the singular locus of angle α is roughly equal to cr^k, with $k = \pm 1 \pm 2p\pi\alpha^{-1}$ or $k = \pm 2p\pi\alpha^{-1}$, where p is an integer.

Using this knowledge one can find a good domain for our operator L when all cone angles are smaller than 2π; namely, one can solve the equation $L\alpha = \phi$ with α, $\nabla\alpha$, and $\mathrm{d}\delta\alpha$ in L^2, and there is uniqueness of such a solution. So what is left to show is that this solution satisfies $\nabla\,\mathrm{d}\alpha \in L^2$. This is true if ϕ is compactly supported: in this case $L\alpha$ equals 0 near the singular locus so one just has to use the preceding computation to prove it. Eventually, one can show that this is also true for arbitrary ϕ by using a sequence ϕ_n, compactly supported and converging to ϕ.

5. Infinitesimal rigidity

Theorem 5.1. *Let M be a hyperbolic cone-manifold whose cone angles are all strictly smaller than 2π. Let h_0 be an infinitesimal Einstein deformation such that h_0 and ∇h_0 are in L^2. Then the deformation h_0 is trivial, that is, there exists a form $\alpha \in \Omega^1 M$ such that $h_0 = \delta^*\alpha$.*

The first step of the proof is to solve the normalization equation $\beta \circ \delta^*\alpha = \beta(h_0)$. By hypothesis h_0 is smooth, square-integrable, with square-integrable covariant derivative, so $\beta(h_0)$ is smooth and in L^2. We can therefore apply Theorem 4.1 to get a solution α of the equation with α, $\nabla\alpha$, $\mathrm{d}\delta\alpha$, and $\nabla\,\mathrm{d}\alpha$ in L^2. Set $h = h_0 - \delta^*\alpha$; our goal is to show that h is actually zero. Notice that we have lost information in this normalization process. For instance, we have no warranty that the covariant derivative of the normalized deformation is still square-integrable; we only have information about some linear combinations of its first order derivative. Returning to the proof: since $E'_g(h_0) = E'_g(h) = 0$, we get

$$\nabla^*\nabla h - 2h + 2(\operatorname{tr} h)g = 0 \quad \text{and} \quad \delta h + \mathrm{d}\operatorname{tr} h = 0.$$

Taking the trace (with respect to g) of the first equation yields $\Delta(\operatorname{tr} h) + 2(n-1)\operatorname{tr} h = 0$. If we integrate against $\operatorname{tr} h$, we get

$$0 = \left\langle \Delta(\operatorname{tr} h) + 2(n-1)\operatorname{tr} h, \operatorname{tr} h \right\rangle = \|\mathrm{d}\operatorname{tr} h\|^2 + 2(n-1)\|\operatorname{tr} h\|^2,$$

so $\operatorname{tr} h = 0$. The above integration by parts is valid, according to Theorem 3.1 and to the fact that $h = h_0 - \delta^*\alpha$ is in L^2, as well as its trace and the differential of its trace $\mathrm{d}\operatorname{tr} h = \mathrm{d}\operatorname{tr} h_0 + \mathrm{d}\delta\alpha$. So what we have now is

$$\nabla^*\nabla h - 2h = 0, \quad \delta h = 0, \quad \text{and} \quad \operatorname{tr} h = 0.$$

We then use the following Weitzenböck formula (cf. [1], Section 12.69), valid for a symmetric 2-tensor h on a hyperbolic manifold:

$$\nabla^*\nabla h = \left(\delta^\nabla \mathrm{d}^\nabla + \mathrm{d}^\nabla \delta^\nabla\right)h + nh - (\operatorname{tr} h)g,$$

where h is viewed as a 1-form with values in T^*M and d^∇ and δ^∇ are the exterior differential and co-differential associated to the connexion ∇ for form with values in T^*M. Combined with the fact that $\delta^\nabla h = \delta h = 0$ and $\operatorname{tr} h = 0$, this formula yields $\delta^\nabla \mathrm{d}^\nabla h + (n-2)h = 0$.

We want to integrate this by parts against h, which is a bit tricky. First we note that $\mathrm{d}^\nabla h$ is in L^2; indeed, ∇h_0 is in L^2, so we just have to check that $\mathrm{d}^\nabla \delta^*\alpha$ is in L^2. But $\mathrm{d}^\nabla \delta^*\alpha = (\mathrm{d}^\nabla)^2\alpha - \frac{1}{2}\mathrm{d}^\nabla \mathrm{d}\alpha$; the operator $(\mathrm{d}^\nabla)^2$ is just minus the curvature operator, so since α and $\nabla\mathrm{d}\alpha$ are L^2, we're done. Then we write $h = h_0 - \delta^*\alpha = h_0 + \frac{1}{2}\mathrm{d}\alpha - \mathrm{d}^\nabla\alpha$; the first part $h_0 + \frac{1}{2}\mathrm{d}\alpha$ is in L^2, with square-integrable covariant derivative, so it can be integrated by parts without any trouble. The second part $\mathrm{d}^\nabla\alpha$ is a bit harder to integrate by parts, but this can be done using again the fact that $(\mathrm{d}^\nabla)^2$ in a zeroth order bounded operator.

Eventually we get

$$0 = \left\langle \delta^\nabla \mathrm{d}^\nabla h + (n-2)h, h\right\rangle = \left\|\mathrm{d}^\nabla h\right\|^2 + (n-2)\|h\|^2,$$

from which we conclude that $h = 0$. So our initial deformation h_0 is equal to $\delta^*\alpha$, that is, it is trivial.

We are now able to prove the infinitesimal rigidity result mentioned in the introduction:

Theorem 5.2. *Let M be a hyperbolic cone-manifold whose cone angles are all strictly smaller than 2π. Then M is infinitesimally rigid among Einstein cone-manifolds with fixed cone angles.*

Indeed, any infinitesimal deformation h of the cone-manifold structure preserving the cone angles is (up to a trivial deformation) in L^2, with covariant derivative ∇h also in L^2, as noticed in section 2. So if it is an Einstein deformation, we can apply the preceding theorem to conclude that any such deformation is trivial.

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